

Final Group Topologies, Kac-Moody Groups and Pontryagin Duality

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Abstract. We study final group topologies and their relations to compactness properties. In particular, we are interested in situations where a colimit or direct limit is locally compact, a k_ω -space, or locally k_ω . As a first application, we show that unitary forms of complex Kac-Moody groups can be described as the colimit of an amalgam of subgroups (in the category of Hausdorff topological groups, and the category of k_ω -groups). Our second application concerns Pontryagin duality theory for the classes of almost metrizable topological abelian groups, resp., locally k_ω topological abelian groups, which are dual to each other. In particular, we explore the relations between countable projective limits of almost metrizable abelian groups and countable direct limits of locally k_ω abelian groups.

Introduction

Given a group G and a family $(f_i)_{i \in I}$ of maps $f_i: X_i \rightarrow G$ from certain topological spaces to G , there exists a finest group topology on G making all of the maps f_i continuous, the so-called *final group topology* with respect to the family $(f_i)_{i \in I}$. Such topologies arise naturally in connection with colimits of topological groups (notably, direct limits), which carry the final group topology with respect to the family of limit maps. Although a final group topology \mathcal{O} always exists, it may be quite elusive in the sense that it may not be clear at all how one could check whether a given subset $U \subseteq G$ belongs to \mathcal{O} .

For example, consider an ascending sequence $G_1 \subseteq G_2 \subseteq \dots$ of topological groups such that all inclusion maps are continuous homomorphisms. Then $G := \bigcup_{n \in \mathbb{N}} G_n$ can be made the direct limit group $\varinjlim G_n$, and one can consider a much more concrete topology \mathcal{T} on G , the *direct limit topology*, defined by declaring $U \subseteq G$ open if and only if $U \cap G_n$ is open in G_n for each $n \in \mathbb{N}$. But unfortunately, \mathcal{T} need not make the group multiplication $G \times G \rightarrow G$ continuous, in which case \mathcal{T} is properly finer than the final group topology \mathcal{O} (see [82]). If this pathology occurs, then the natural continuous bijection

$$\varinjlim (G_n \times G_n) \rightarrow \varinjlim G_n \times \varinjlim G_n \quad (1)$$

is not a homeomorphism (cf. [45]), and thus the pathology is related to the non-compatibility of direct products and direct limit topologies.

However, both pathologies disappear if each G_n is locally compact: Then the spaces in (1) coincide, and $\mathcal{T} = \mathcal{O}$ (see, e.g., [45, Theorem 4.1]).

Classification: 22A05 (main); 18A30, 20E06, 22C05, 22D35, 22E15, 43A40, 51B25, 51E24, 54D50

Keywords: Final group topology, colimit, direct limit, inductive limit, projective limit, inverse limit, amalgam, locally compact group, k_ω -space, locally k_ω space, k_ω -group, reflexive group, almost metrizable group, Čech complete group, Pontryagin duality, Kac-Moody group, building, twin building, Phan system

In this article, we discuss more comprehensive classes of topological groups (and spaces), for which the preceding difficulties cannot occur. Although the topological groups considered need not be locally compact, we obtain results concerning final group topologies, colimits and direct limits for such groups, which involve compactness in less direct ways. In particular, we obtain results for topological groups whose underlying topological spaces are k_ω or locally k_ω .

Recall that a topological space is called a k_ω -space if it is the direct limit of an ascending sequence of compact (Hausdorff) subspaces. The class of k_ω -spaces is quite general and comprises, for example, all countable CW-complexes [73, §5] and countable direct limits of σ -compact locally compact groups (cf. [27], [82]); yet it is restrictive enough to allow a systematic treatment. In the theory of topological groups, k_ω -spaces have attracted considerable interest because free topological groups over compact (or k_ω -) spaces are k_ω -spaces (see [35, Theorem 4], [63, Corollary 1] and [74, Theorem 5.2]), and these are essentially the only examples of free topological groups whose topology is well accessible. The situation is similar for Hausdorff quotient groups of such free groups (which are k_ω -spaces as well). These subsume important categorical constructs, like free products of two k_ω -groups with (or without) amalgamation ([54]–[57], [60]).

The class of locally k_ω spaces (introduced below) provides a common roof for the classes of locally compact spaces on the one hand, and k_ω -spaces on the other.

Our motivation to study k_ω -spaces stems from two specific sources of examples: On the one hand, these are examples from (infinite-dimensional) topological geometry and geometric group theory, more precisely Moufang topological twin buildings (cf. [42]) and real and complex Kac-Moody groups including their non-split forms and certain twisted variants (cf. [51], [86] and [78], respectively). On the other hand, k_ω -groups arise most naturally in the duality theory of topological abelian groups, because dual groups of metrizable abelian groups are k_ω [6, Corollary 4.7].

The logical structure of the article guarantees that readers interested in only one of the two fields of applications can skip the discussion of the other.

Local compactness of colimits. After a preparatory section, in Section 2 we study situations ensuring that a final group topology is locally compact (Proposition 2.2). For instance, we consider a diagram of countably many σ -compact locally compact groups and its colimit G in the category of abstract groups. If G admits a locally compact group topology making the limit maps continuous, then this topology makes G the colimit in the category of topological groups (Corollary 2.3). We remark that results concerning local compactness of colimits for special diagrams are available in the literature, notably for free products $A * B$ and free products $A *_C B$ with amalgamation. In these cases, local compactness usually rules out interesting situations.¹ The picture changes if one consid-

¹For example, for A, B non-trivial, local compactness of $A * B$ necessitates that A and B are discrete (and likewise for infinite free products, see [67, Corollary 1]). Furthermore, if C is a proper subgroup of

ers more complicated diagrams of topological groups and their colimits (e.g., colimits of amalgams of more than two factors). Then a rich supply of non-trivial examples becomes available, as we hope to illustrate by results concerning compact Lie groups in Section 3.

We mention that a typical feature of our amalgamation results becomes visible at this point: While topological group theoreticians mainly studied simple types of amalgams of quite general groups, our geometrically motivated examples involve complicated amalgams of rather special groups.

Lie groups determined by an amalgam of subgroups. Amalgams and their universal enveloping groups (i.e., their colimits in the category of groups) play an important role in group theory. The possibly most famous result in this area, the Curtis-Tits Theorem, asserts that the universal version of an arbitrary Chevalley group of (twisted) rank at least three is the universal enveloping group of the amalgam consisting of the groups $\langle X_{\pm\alpha} \rangle$ and $\langle X_{\pm\alpha}, X_{\pm\beta} \rangle$, where α, β run through a fundamental system of roots corresponding to the Chevalley group and X_α denotes the respective root subgroup (cf. [21], [83], also [33, Theorem 2.9.3]). A similar result has been proved for compact Lie groups (cf. [15]), and for complex Kac-Moody groups and their unitary forms (cf. [51]), as abstract groups.

A lemma by Tits (cf. [85]; restated as Lemma 3.3 in the present article) ties the above amalgamation results to the theory of simplicial complexes and, in particular, to the theory of Tits buildings and twin buildings. For compact Lie groups and unitary forms of complex Kac-Moody groups, this relies on the fact that a Tits building of rank at least three is simply connected (cf. [84]). For Chevalley groups and Kac-Moody groups, it relies on the fact that the opposites geometry of the corresponding twin building is simply connected (cf. [69]). Recently, a machinery was developed to classify the corresponding amalgams, thus improving the above-mentioned results, cf. [13], [22], [36], [37].

The amalgamation results from [37] treated compact Lie groups as abstract groups. In Section 3, we transfer them into the setting of locally compact groups (and Lie groups), using results from Section 2. All relevant mappings identified earlier as isomorphisms of abstract groups turn out to be topological isomorphisms (see Theorems 3.1 and 6.13 and their respective proofs).

Locally k_ω spaces. In Section 4, we introduce locally k_ω spaces (Hausdorff spaces each point of which has an open neighbourhood which is a k_ω -space). These form a quite natural class of topological spaces, which subsumes all k_ω -spaces and all locally compact spaces (and is therefore prone to unify results known for the two subclasses). Given an ascending sequence $X_1 \subseteq X_2 \subseteq \dots$ of locally k_ω spaces with continuous inclusion maps $X_n \rightarrow X_{n+1}$, we show that the direct limit topology on $X = \bigcup_{n \in \mathbb{N}} X_n$ is locally k_ω (Proposition 4.5). Furthermore, given another such sequence $Y_1 \subseteq Y_2 \subseteq \dots$, with direct limit Y , we show that

both A and B , then local compactness of $A *_C B$ forces that C is open in $A *_C B$ (see [3, Corollary 3], or [60, Theorem 3] for the case where C is normal in A and B). It is also known that an amalgamated product $A *_C B$, with C a proper subgroup of A and B , cannot be compact [2].

the product topology on $X \times Y$ makes $X \times Y$ the direct limit topological space $\varinjlim X_n \times Y_n$ (Proposition 4.7). The compatibility of direct limits and direct products was known previously for ascending sequences of locally compact spaces (see, e.g., [45, Theorem 4.1] and [27, Proposition 3.3]). It is needed to get a grip on direct limits of topological groups [27], [45] (as already indicated) and direct limits of Lie groups (see [27]–[29], [70] and [71]).

Locally k_ω groups and their direct limits. In Section 5, we show that a topological group is locally k_ω if and only if it has an open subgroup which is a k_ω -group (Proposition 5.3). We also prove that, for each ascending sequence $G_1 \leq G_2 \leq \dots$ of locally k_ω groups with continuous inclusion maps, the direct limit topology on $G := \bigcup_{n \in \mathbb{N}} G_n$ is locally k_ω and makes G a topological group (Proposition 5.4). This result provides a common generalization of known facts concerning direct limits of locally compact groups ([27, Corollary 3.4], [82, Theorem 2.7]) and abelian k_ω -groups [4, Proposition 2.1]. We also formulate a condition ensuring that a final group topology is k_ω (Proposition 5.8). This entails the existence of certain colimits of k_ω -groups (Corollary 5.10). For closely related results concerning the group topology generated by a subspace X which is compact or k_ω , see [63], [66] and [72].

Kac-Moody groups determined by an amalgam of subgroups. The topological amalgamation results discussed in Section 3 have an infinite-dimensional analogue which we describe in Section 6. On the algebraic level, unitary forms of complex Lie groups are replaced by unitary forms of complex Kac-Moody groups; on the geometric level, spherical buildings are replaced by twin buildings; and on the topological level, (locally) compact spaces are replaced by k_ω -spaces. While the proofs become more technical, the results (cf. Theorem 6.12) are analogous to the finite-dimensional case.

The duality of locally k_ω , resp., almost metrizable abelian groups. Sections 7 and 8 are devoted to the duality theory of (Hausdorff) topological abelian groups. Given an abelian Hausdorff group G , let G^* be its *dual group*, i.e., the group of all continuous homomorphisms from G to the circle group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, equipped with the compact-open topology. It is known that G^* is a k_ω -group if G is metrizable (see [6, Corollary 4.7]). Conversely, G^* is metrizable (and complete) for each topological abelian group G which is a k_ω -space by [6, Propositions 2.8 and 4.11]. The duality of metrizable groups and k_ω -groups was explored in [4], in particular the duality between countable direct limits of k_ω -groups and countable projective limits of metrizable groups. Our goal is to generalize the results from [4] to the larger classes of almost metrizable abelian groups and locally k_ω abelian groups. Also these are in duality: Exploiting the fact that a topological abelian group G is almost metrizable (as in [75] or [6, Definition 1.22]) if and only if it has a compact subgroup K such that G/K is metrizable (see [6, Proposition 2.20] or [75]), we show in Section 7 that G^* is locally k_ω for almost metrizable G , while G^* is almost metrizable whenever G is locally k_ω (Proposition 7.1).

Pontryagin duality for direct and projective limits. In Section 8, we study the dual groups of countable direct limits of locally k_ω abelian groups, and the dual groups of countable projective limits of almost metrizable abelian groups. Since locally k_ω groups subsume both locally compact abelian groups and abelian k_ω -groups, our results provide a common generalization of Kaplan’s classical results concerning the duality of countable direct (and projective) limits of locally compact abelian groups ([52], [53]) and the recent studies from [4] just mentioned. In the special cases of complete metrizable groups and k_ω -groups treated in [4], we are able to loosen some of the hypotheses of [4] and thus to strengthen their results. Cf. [5] and [11] for complementary studies of the continuous duality of direct and projective limits of convergence groups.

Acknowledgements. The authors thank Andreas Mars (Darmstadt) and an anonymous referee for comments on a preliminary version of the article. The manuscript was partially written while the second author held a substitute professorship at the TU Braunschweig. He would like to express his gratitude to the members of the Institute of Computational Mathematics of the TU Braunschweig for their hospitality. The first and second author acknowledge support by the German Research Foundation (DFG) (projects: GL 357/2-1 and GL 357/5-1, respectively, GR 2077/5-1 and GR 2077/7-1). During the revision of this article, the third author was partially supported by Swiss National Science Foundation (SNF), grant PP002-102765.

1 Notation and terminology

Part of our terminology has already been described in the introduction. We now fix further terminology and notation. In the following, \mathbb{G} denotes the category of groups and homomorphisms. All compact (or locally compact) spaces are assumed Hausdorff. The category of all (not necessarily Hausdorff) topological groups and continuous homomorphisms will be denoted by \mathbf{TG} , and \mathbf{HTG} , \mathbf{LCG} and \mathbf{LIE} are its full subcategories of Hausdorff topological groups, locally compact topological groups and finite-dimensional real Lie groups, respectively. Surjective, open, continuous homomorphisms between topological groups shall be referred to as *quotient morphisms*. Finally, we shall use the term “locally convex space” as an abbreviation for “locally convex topological vector space.”

Diagrams and colimits. Recall that a category \mathbb{I} is called *small* if its class $\mathbf{ob}(\mathbb{I})$ of objects is a set. A *diagram* in a category \mathbb{A} is a covariant functor $\delta: \mathbb{I} \rightarrow \mathbb{A}$ from a small category \mathbb{I} to \mathbb{A} . A *cone* over a diagram $\delta: \mathbb{I} \rightarrow \mathbb{A}$ is a natural transformation $\varphi: \delta \rightarrow \gamma$ to a “constant” diagram γ , i.e., to a diagram $\gamma: \mathbb{I} \rightarrow \mathbb{A}$ such that $G := \gamma(i) = \gamma(j)$ for all $i, j \in \mathbf{ob}(\mathbb{I})$ and $\gamma(\alpha) = \text{id}_G$ for each morphism α in \mathbb{I} . We can think of a cone as the object $G \in \mathbf{ob}(\mathbb{A})$, together with the family $(\varphi(i))_{i \in \mathbf{ob}(\mathbb{I})}$ of morphisms $\varphi(i): \delta(i) \rightarrow G$, such that

$$\varphi(j) \circ \delta(\alpha) = \varphi(i) \quad \text{for all } i, j \in \mathbf{ob}(\mathbb{I}) \text{ and } \alpha \in \mathbf{Mor}(i, j).$$

A cone $(G, (\varphi_i)_{i \in \mathbf{ob}(\mathbb{I})})$ is called a *colimit* of δ if, for each cone $(H, (\psi_i)_{i \in \mathbf{ob}(\mathbb{I})})$, there is a unique morphism $\psi: G \rightarrow H$ such that $\psi \circ \varphi_i = \psi_i$ for all $i \in \mathbf{ob}(\mathbb{I})$. If it exists, a colimit

is unique up to natural isomorphism. We expect that the reader is familiar with the most basic types of colimits and corresponding diagrams, like direct limits (corresponding to the case where \mathbb{I} is a directed set), free products of groups, and amalgamated products $A *_C B$.

Colimits of topological groups. Standard arguments show that colimits exist for any diagram in \mathbb{G} , \mathbb{TG} and \mathbb{HTG} (as is well known). Given a diagram $\delta: \mathbb{I} \rightarrow \mathbb{TG}$ of topological groups, with colimit $(G, (\lambda_i)_{i \in \text{ob}(\mathbb{I})})$ in the category \mathbb{G} of abstract groups, there is a finest group topology \mathcal{O} on G making each λ_i continuous; then $((G, \mathcal{O}), (\lambda_i)_{i \in \text{ob}(\mathbb{I})})$ is the colimit of δ in \mathbb{TG} . If δ is a diagram in \mathbb{HTG} here, then $H := G/\overline{\{1\}}$ is a Hausdorff group and $(H, (q \circ \lambda_i)_{i \in \text{ob}(\mathbb{I})})$ is readily seen to be the colimit of δ in \mathbb{HTG} , where $q: G \rightarrow G/\overline{\{1\}} = H$ is the canonical quotient morphism. Frequently, the final group topology on G is Hausdorff, in which case the \mathbb{HTG} -colimit is obtained by topologizing the colimit of abstract groups. For example, this situation occurs in the case of free products of Hausdorff topological groups [34] and amalgamated products $A *_C B$ with C a closed central subgroup of A and B (see [59]); if A and B are k_ω -groups, it suffices that C is a closed normal subgroup [55] or compact [56]. This additional information is not available through general category-theoretical arguments, but requires a solid amount of work.

Amalgams and their enveloping groups. In the article [37] to be extended in Section 3, amalgams of groups play a central role, which were defined there as families $(H_i)_{i \in J}$ of groups satisfying certain axioms, ensuring in particular that $H_i \cap H_j$ is a subgroup of H_i and H_j , for all $i, j \in J$ (see [37, Definition 2.5]). In the present paper, we prefer a (wider) definition of amalgams in category-theoretical terms:

A diagram $\delta: \mathbb{I} \rightarrow \mathbb{G}$ of groups is called an *amalgam* if \mathbb{I} is the category associated with some partially ordered set (J, \leq) and $\delta(\alpha)$ is a monomorphism of groups for each morphism α in \mathbb{I} . The first condition means that $\text{ob}(\mathbb{I}) = J$, and furthermore for $a, b \in J$ there exists one (and only one) morphism $a \rightarrow b$ if and only if $a \leq b$. (Sometimes we prefer to reverse the partial order on J to the effect that there exists a morphism $a \rightarrow b$ if and only if $a \geq b$). Following the terminology in [38], a cone $(G, (\varphi_i)_{i \in \text{ob}(\mathbb{I})})$ over δ in \mathbb{G} will also be called an *enveloping group* of the amalgam and its colimit a *universal enveloping group*. Analogous terminology will be used for amalgams of topological groups. We would like to point out that in [31], [37] and other references, the terminology “completion” and “universal completion” has been used. This terminology, however, is unfortunate, because an amalgam of topological groups might have a universal enveloping group which is not a complete topological group (in its left uniform structure). As it may sometimes be useful in this case to pass to the completed topological group, the clash of terminology might lead to confusion.

Of particular importance for us are amalgams $\delta: \mathbb{I} \rightarrow \mathbb{G}$, where J is a set and \mathbb{I} the small category whose set of objects is $\text{ob}(\mathbb{I}) = \binom{J}{1} \cup \binom{J}{2}$, and such that, besides identity morphisms, there is exactly one morphism $\{i\} \rightarrow \{i, j\}$ and $\{j\} \rightarrow \{i, j\}$ for all distinct $i, j \in J$. We can think of such an amalgam as a family of groups $(G_i)_{i \in \text{ob}(\mathbb{I})}$ (where $G_{\{i\}} := \delta(\{i\})$, $G_{\{i, j\}} := \delta(\{i, j\})$) together with monomorphisms of groups $G_{\{i\}} \rightarrow G_{\{i, j\}}$,

resp., $G_{\{j\}} \rightarrow G_{\{i,j\}}$ (cf. also [40, Definition 2.15] for this type of amalgam).

Basic facts concerning direct limits of topological spaces. For basic facts concerning direct limits of topological spaces and topological groups, the reader is referred to [27], [41], [45] and [82] (where also many of the pitfalls and subtleties of the topic are described). In particular, we shall frequently use that the set underlying a direct limit of groups is the corresponding direct limit in the category of sets, and that the direct limit of a direct system of topological spaces in the category of (not necessarily Hausdorff) topological spaces is its direct limit in the category of sets, equipped with the final topology with respect to the limit maps. For later reference, we compile further simple facts in a lemma:

Lemma 1.1 *Let $((X_i)_{i \in I}, (f_{ij})_{i \geq j})$ be a direct system in the category of topological spaces and continuous maps, with direct limit $(X, (f_i)_{i \in I})$.*

- (a) *If $U_i \subseteq X_i$ is open and $f_{ij}(U_j) \subseteq U_i$ for $i \geq j$, then $\bigcup_{i \in I} f_i(U_i)$ is open in X , and $U = \varinjlim U_i$ holds for the induced topology.*
- (b) *If $Y \subseteq X$ is closed or open, then $Y = \varinjlim f_i^{-1}(Y)$ as a topological space.*
- (c) *If each f_{ij} is injective, then each f_i is injective. If each f_{ij} is a topological embedding, then each f_i is a topological embedding.*
- (d) *If I is countable, each X_i is T_1 and each f_i is injective, then each compact subset K of X is contained in $f_i(X_i)$ for some $i \in I$. If, furthermore, each f_i is a topological embedding, then $K = f_i(L)$ for some $i \in I$ and some compact subset $L \subseteq X_i$.*

Proof. (a) is quite obvious (cf. [27, Lemma 3.1]). Part (b) follows from the definition of final topologies. The injectivity in (c) is clear from the construction of X (see, e.g., [27, §2]). For the second assertion, see [71, Lemma A.5]. The first part of (d) is [28, Lemma 1.7(d)], and its second part follows from (c) (see also [41, Lemma 2.4]). \square

If all bonding maps $f_{ij}: X_j \rightarrow X_i$ are topological embeddings, then a direct system of topological spaces (or topological groups) is called *strict*.

2 Local compactness and final group topologies

We describe a condition ensuring that a final group topology is locally compact. A simple version of the Open Mapping Theorem will be used.

Lemma 2.1 *Let $f: G \rightarrow H$ be a surjective, continuous homomorphism between Hausdorff topological groups. If G is σ -compact and H is a Baire space, then f is open and H is locally compact.*

Proof. By hypothesis, $G = \bigcup_{n \in \mathbb{N}} K_n$ for certain compact sets $K_n \subseteq G$ and thus $H = \bigcup_{n \in \mathbb{N}} f(K_n)$ with $f(K_n)$ compact. Since H is a Baire space, $f(K_n)$ has non-empty interior

for some $n \in \mathbb{N}$, and thus H is locally compact. Like any continuous surjection between compact Hausdorff spaces, $f|_{K_n}: K_n \rightarrow f(K_n)$ is a quotient map. Let $q: G \rightarrow G/\ker(f)$ be the quotient homomorphism and $\varphi: G/\ker(f) \rightarrow H$ be the bijective continuous homomorphism induced by f . Then $\varphi^{-1} \circ f|_{K_n} = q|_{K_n}$ is continuous, whence $\varphi^{-1}|_{f(K_n)}$ is continuous, φ^{-1} is a continuous homomorphism, and φ is a topological isomorphism. \square

Proposition 2.2 *Let G be a Hausdorff topological group and $(f_i)_{i \in I}$ be a countable family of continuous maps $f_i: X_i \rightarrow G$, such that X_i is σ -compact for each $i \in I$ and $\bigcup_{i \in I} f_i(X_i)$ generates G . Then G is locally compact if and only if G is a Baire space. In this case, the given locally compact topology on G is the final group topology with respect to $(f_i)_{i \in I}$.*

Proof. Assume that G is Baire (which follows from local compactness). The final group topology \mathcal{T} being finer than the given topology \mathcal{O} , it is Hausdorff. Since (G, \mathcal{T}) is generated by its σ -compact subset $\bigcup_{i \in I} f_i(X_i)$, the topological group (G, \mathcal{T}) is σ -compact. As (G, \mathcal{O}) is Baire, Lemma 2.1 shows that the continuous surjective homomorphism $(G, \mathcal{T}) \rightarrow (G, \mathcal{O})$, $g \mapsto g$ is an isomorphism of topological groups and (G, \mathcal{O}) is locally compact. \square

The following special case (applied to amalgams of finitely many Lie groups) will be used presently to transfer the purely algebraic considerations from [37] into a topological (and Lie theoretic) context.

The setting is as follows: $\delta: \mathbb{I} \rightarrow \mathbb{LCG}$ is a diagram of σ -compact locally compact groups $G_i := \delta(i)$ for $i \in I := \text{ob}(\mathbb{I})$ and continuous homomorphisms $\varphi_\alpha := \delta(\alpha): G_i \rightarrow G_j$ for $i, j \in I$ and $\alpha \in \text{Mor}(i, j)$, such that I is countable. Furthermore, $(G, (\lambda_i)_{i \in I})$ is a colimit of the diagram δ in the category of abstract groups, with homomorphisms $\lambda_i: G_i \rightarrow G$.

Corollary 2.3 *If there exists a locally compact Hausdorff group topology \mathcal{O} on G making $\lambda_i: G_i \rightarrow (G, \mathcal{O})$ continuous for each $i \in I$, then $((G, \mathcal{O}), (\lambda_i)_{i \in I})$ is a colimit of δ in the category of topological groups, in the category of Hausdorff groups, and in the category of locally compact groups. In particular, there is only one such locally compact group topology on G . If each G_i is a σ -compact Lie group and (G, \mathcal{O}) is a Lie group, then $((G, \mathcal{O}), (\lambda_i)_{i \in I})$ also is a colimit of δ in the category \mathbb{LIE} of Lie groups. \square*

3 Lie groups determined by rank two subgroups

We are now in the position to formulate a topological variant of the main results of [15].

Let G be a simply connected compact semisimple Lie group, let T be a maximal torus of G , let $\Sigma = \Sigma(G_{\mathbb{C}}, T_{\mathbb{C}})$ be its root system, and let Π be a system of fundamental roots of Σ . To each root $\alpha \in \Pi$ corresponds some compact semisimple Lie group $K_\alpha \leq G$ of rank one such that T normalizes K_α . For simple roots α, β , we denote by $K_{\alpha\beta}$ the group generated by the groups K_α and K_β , and by $\Sigma_{\alpha\beta}$ its root system relative to the torus $T_{\alpha\beta} = T \cap K_{\alpha\beta}$. The group $K_{\alpha\beta}$ is a compact semisimple Lie group of rank two and $\{\alpha, \beta\}$

is a fundamental system of $\Sigma_{\alpha\beta}$. The pair (K_α, K_β) is called a *standard pair* of $K_{\alpha\beta}$, as is any image of (K_α, K_β) under a homomorphism from $K_{\alpha\beta}$ onto a central quotient of $K_{\alpha\beta}$. Standard pairs are conjugate. In fact, maximal tori are conjugate (cf. [47, Theorem 6.25]), and, if $\alpha, \beta \in \Pi$ and $\alpha_1, \beta_1 \in \Sigma$ have the same lengths and the same angle, there exists an element w of the Weyl group with $w(\alpha_1) = \alpha$ and $w(\beta_1) = \beta$ (cf. [16]).

Then the following topological version of Borovoi's theorem [15] holds:

Theorem 3.1 *Let G be a simply connected compact semisimple Lie group with Lie group topology \mathcal{O} , let T be a maximal torus of G , let $\Sigma = \Sigma(G_{\mathbb{C}}, T_{\mathbb{C}})$ be its root system, and let Π be a system of fundamental roots of Σ . Let \mathbb{I} be a small category with objects $\binom{\Pi}{1} \cup \binom{\Pi}{2}$ and morphisms $\{\alpha\} \rightarrow \{\alpha, \beta\}$, for all $\alpha, \beta \in \Pi$, and let $\delta: \mathbb{I} \rightarrow \mathbb{L}\mathbb{C}G$ be a diagram with $\delta(\{\alpha\}) = K_\alpha$, $\delta(\{\alpha, \beta\}) = K_{\alpha\beta}$, and $\delta(\{\alpha\} \rightarrow \{\alpha, \beta\}) = (K_\alpha \hookrightarrow K_{\alpha\beta})$.*

Then $((G, \mathcal{O}), (\iota_i)_{i \in \binom{\Pi}{1} \cup \binom{\Pi}{2}})$, where ι_i is the natural inclusion map, is a colimit of δ in the category $\mathbb{L}\mathbb{I}\mathbb{E}$ of Lie groups.

Before we prove the theorem, we remind the reader of the notion of a Tits building and restate Tits' Lemma.

Definition 3.2 Following [84, Theorem 5.2], see also [26, Section 2] and [14, Section 6.8], the *Tits building* $\Delta(G, \mathbb{F})$ of G over \mathbb{F} , where G is a reductive algebraic group defined over a field \mathbb{F} , consists of the simplicial complex whose simplices are indexed by the \mathbb{F} -parabolic subgroups of G ordered by the reversed inclusion relation on the parabolic subgroups.

Lemma 3.3 (Tits' Lemma; cf. [85]) *Let A be a poset, let G be a group acting on A , let F be a fundamental domain for the action of G on A , i.e., (i) $a \in F$ and $b \leq a$ implies $b \in F$, (ii) $A = G.F$, (iii) $G.a \cap F = \{a\}$ for all $a \in F$, and let \mathbb{I} be a small category with objects F and morphisms $b \leftarrow a$ for all $b \leq a \in F$. Then the poset A is simply connected if and only if G is the colimit of the diagram $\delta: \mathbb{I} \rightarrow \mathbb{G}$, where $\delta(a) = G_a$ and $\delta(b \leftarrow a) = (G_b \hookrightarrow G_a)$.*

Here a poset A is called simply connected if the associated simplicial complex $|A|$ – with the non-empty finite chains of A as simplices – is simply connected.

Proof of Theorem 3.1. For Lie rank $|\Pi|$ of G at most two there is nothing to show, so we can assume that $|\Pi| \geq 3$. Let $G_{\mathbb{C}}$ be the complexification of G and let Δ be the Tits building of $G_{\mathbb{C}}$ (over \mathbb{C}), which is a $(|\Pi| - 1)$ -dimensional simplicial complex. Let B be a Borel subgroup, i.e., a minimal parabolic subgroup, of $G_{\mathbb{C}}$, containing the maximal torus T . The $(|\Pi| - 1)$ -simplex σ of Δ corresponding to B is a fundamental domain for the conjugation action of $G_{\mathbb{C}}$ on Δ (see [84, Theorem 3.2.6], also [26, Section 1], [81, Chapter 6]). By the Iwasawa decomposition (see [43, Theorem VI.5.1] or [44, Theorem III.6.32]), we have $G_{\mathbb{C}} = GB$, so σ is also a fundamental domain for the conjugation action of G on Δ . The stabilizers in G of the sub-simplices codimension one and codimension two of σ are exactly the groups $K_\alpha T$ and $K_{\alpha\beta} T$. By the simple connectedness of Tits buildings of rank at least three (cf. [17, Theorem IV.5.2], [84, Theorem 13.32]; also [26, Theorem 2.2]) plus

an induction on $|\Pi|$ applied to Lemma 3.3, the group G is a colimit — in the category of abstract groups — of the diagram $\delta': \mathbb{I} \rightarrow \mathbb{LCCG}$ with $\delta'(\{\alpha\}) = K_\alpha T$, $\delta'(\{\alpha, \beta\}) = K_{\alpha\beta} T$, and $\delta'(\{\alpha\} \rightarrow \{\alpha, \beta\}) = (K_\alpha T \hookrightarrow K_{\alpha\beta} T)$. By [32, Lemma 29.3] (or by a reduction argument as in the proof of [39, Theorem 2] or in the proof of [36, Theorem 4.3.6]), the torus T can be reconstructed from the rank two tori $T_{\alpha\beta}$, $\alpha, \beta \in \Pi$, and so the group G in fact is a colimit — in the category of abstract groups — of the diagram δ . Application of Corollary 2.3 to δ yields the claim. \square

4 Basic facts on k_ω -spaces and locally k_ω spaces

In this section, we introduce locally k_ω spaces and compile various useful properties of k_ω -spaces and locally k_ω spaces, for later use. In particular, we discuss direct limits in the category of locally k_ω spaces.

Definition 4.1 A Hausdorff topological space X is a k_ω -space if there exists an ascending sequence of compact subsets $K_1 \subseteq K_2 \subseteq \cdots \subseteq X$ such that $X = \bigcup_{n \in \mathbb{N}} K_n$ and $U \subseteq X$ is open if and only if $U \cap K_n$ is open in K_n for each $n \in \mathbb{N}$ (i.e., $X = \varinjlim K_n$ as a topological space). Then $(K_n)_{n \in \mathbb{N}}$ is called a k_ω -sequence for X . We say that a Hausdorff topological space X is *locally k_ω* if each point has an open neighbourhood which is a k_ω -space in the induced topology.

Thus every k_ω -space is locally k_ω . It is clear that every locally k_ω -space X is a k -space (i.e., X is Hausdorff and a subset $U \subseteq X$ is open if and only if $U \cap K$ is open in K for each compact set $K \subseteq X$).

Proposition 4.2 (a) σ -compact locally compact spaces are k_ω .

- (b) Closed subsets of k_ω -spaces are k_ω in the induced topology.
- (c) Finite products of k_ω -spaces are k_ω in the product topology.
- (d) Hausdorff quotients of k_ω -spaces are k_ω .
- (e) Countable disjoint unions of k_ω -spaces are k_ω .
- (f) Every locally compact space is locally k_ω .
- (g) Open subsets of locally k_ω spaces are locally k_ω .
- (h) Closed subsets of locally k_ω spaces are locally k_ω in the induced topology.
- (i) Finite products of locally k_ω spaces are locally k_ω in the product topology.
- (j) Hausdorff quotients of locally k_ω spaces under open quotient maps are locally k_ω .

Proof. (a)–(e) are well known: see, e.g., [24] and the references therein.² Part (h) follows from (b), (i) from (c), and (j) from (d).

(f) If X is locally compact and $x \in X$, choose compact subsets K_1, K_2, \dots such that $x \in K_1$ and K_{n+1} contains K_n in its interior. Then $U := \bigcup_{n \in \mathbb{N}} K_n$ is an open neighbourhood of x in X and U is a k_ω -space with $(K_n)_{n \in \mathbb{N}}$ as a k_ω -sequence.

(g) It suffices to show that each open neighbourhood W of a point x in a k_ω -space X contains an open neighbourhood U of x which is a k_ω -space. To see this, let $(K_n)_{n \in \mathbb{N}}$ be a k_ω -sequence for X , with $x \in K_1$. Then $K_1 \cap W$ contains a compact neighbourhood C_1 of x in K_1 . Recursively, we find a compact subsets $C_n \subseteq W \cap K_n$ such that C_n is a neighbourhood of C_{n-1} in $K_n \cap W$, for each $n \in \mathbb{N}$. Let U_n be the interior of C_n in K_n . Then $U := \bigcup_{n \in \mathbb{N}} U_n \subseteq W$ is open in X and $U = \varinjlim U_n$ in the induced topology, by Lemma 1.1 (a). Since $U_n \subseteq C_n \subseteq U_{n+1}$ for each $n \in \mathbb{N}$, we see that $U = \varinjlim U_n = \varinjlim C_n$ is a k_ω -space. \square

Proposition 4.2 (j) ensures that every Hausdorff quotient group of a locally k_ω group is locally k_ω .

Lemma 4.3 *Let X be a locally k_ω space and $Y \subseteq X$ be a σ -compact subset. Then Y has an open neighbourhood U in X which is a k_ω -space.*

Proof. Since Y is σ -compact, we find a sequence $(U_n)_{n \in \mathbb{N}}$ of open subsets of X such that $Y \subseteq \bigcup_{n \in \mathbb{N}} U_n =: U$ and U_n is a k_ω -space, for each $n \in \mathbb{N}$. Then U is a Hausdorff quotient of the k_ω -space $\coprod_{n \in \mathbb{N}} U_n$ and therefore a k_ω -space, by Proposition 4.2 (d) and (e). \square

Recall that a map $f: X \rightarrow Y$ between topological spaces is called *compact-covering* if for each compact subset $K \subseteq Y$, there exists a compact set $L \subseteq X$ such that $K \subseteq f(L)$.

Lemma 4.4 *Every quotient map $q: X \rightarrow Y$ between k_ω -spaces is compact-covering.*

Proof. Let $K \subseteq Y$ be compact. If $(K_n)_{n \in \mathbb{N}}$ is a k_ω -sequence for X , then $(q(K_n))_{n \in \mathbb{N}}$ is a k_ω -sequence for Y and thus $K \subseteq q(K_n)$ for some n , by Lemma 1.1 (d). \square

Proposition 4.5 *Let $X_1 \subseteq X_2 \subseteq \dots$ be a sequence of k_ω -spaces (resp., locally k_ω spaces) X_n , such that the inclusion map $X_n \rightarrow X_{n+1}$ is continuous, for each $n \in \mathbb{N}$. Then the final topology \mathcal{O} on $X := \bigcup_{n \in \mathbb{N}} X_n$ with respect to the inclusion maps $X_n \rightarrow X$ is k_ω (resp., locally k_ω), and makes X the direct limit $X = \varinjlim X_n$ in each of the categories of topological spaces, Hausdorff topological spaces, and k_ω -spaces (resp., locally k_ω spaces).*

²For the reader's convenience, we mention: In (a), any exhaustion $K_1 \subseteq K_2 \subseteq \dots$ of the space by compact sets with K_n in the interior of K_{n+1} provides a k_ω -sequence. Part (b) is a special case of Lemma 1.1 (b). Part (c) follows from Proposition 4.7 below. Part (d) follows from the transitivity of final topologies. Part (e): If $X = \coprod_{j \in \mathbb{N}} X_j$, where each X_j is a k_ω -space with a k_ω -sequence $(K_n^{(j)})_{n \in \mathbb{N}}$, then clearly $K_n := \bigcup_{j \leq n} K_n^{(j)}$ defines a k_ω -sequence for X .

Proof. If each X_j is a k_ω -space, let $(K_n^{(j)})_{n \in \mathbb{N}}$ be a k_ω -sequence for X_j . After replacing $K_n^{(j)}$ with $\bigcup_{i=1}^j K_n^{(i)}$, we may assume that $K_n^{(i)} \subseteq K_n^{(j)}$ for all $i, j \in \mathbb{N}$ such that $i \leq j$. Define $K_n := K_n^{(n)}$ for $n \in \mathbb{N}$. Since each K_n is compact and the inclusion maps $K_n \rightarrow K_{n+1}$ are topological embeddings, the final topology \mathcal{T} on X with respect to the inclusion maps $K_n \rightarrow X$ is Hausdorff (see [27, Proposition 3.6 (a)]). Thus $(X, \mathcal{T}) = \varinjlim K_n$ is a k_ω -space. The inclusion maps $K_n \rightarrow X_n \rightarrow (X, \mathcal{O})$ being continuous, we deduce that $\mathcal{O} \subseteq \mathcal{T}$. To see the converse, let $U \in \mathcal{T}$. Then $U \cap K_n$ is open in K_n for each $n \in \mathbb{N}$. Given $n \in \mathbb{N}$ and $j \in \mathbb{N}$, let $m := \max\{n, j\}$. Then $U \cap K_n^{(j)} = (U \cap K_m) \cap K_n^{(j)}$ is open in $K_n^{(j)}$, the inclusion map $K_n^{(j)} \rightarrow K_m$ being continuous. Hence $U \cap X_j$ is open in X_j for each j and thus $U \in \mathcal{O}$. Hence $\mathcal{O} = \mathcal{T}$, as required.

In the case of locally k_ω spaces X_n , let $x \in X$, say $x \in X_1$ (after passing to a cofinal subsequence). Let $U_1 \subseteq X_1$ be an open neighbourhood of x which is a k_ω -space. Using Lemma 4.3, we find open subsets $U_n \subseteq X_n$ which are k_ω -spaces and such that $U_n \subseteq U_{n+1}$, for each $n \in \mathbb{N}$. Then $U := \bigcup_{n \in \mathbb{N}} U_n$ is an open subset of X (see Lemma 1.1 (a)) and $U = \varinjlim U_n$ is a k_ω -space, by what has just been shown. It only remains to show that X is Hausdorff. To this end, suppose that $x, y \in X$ are distinct elements. We may assume that $x, y \in X_1$. The set $\{x, y\}$ being compact, it has an open neighbourhood $U_1 \subseteq X_1$ which is a k_ω -space (by Lemma 4.3). Choose $U_n \subseteq X_n$ as just described. Then both x and y are contained in the open subset $U = \bigcup_{n \in \mathbb{N}} U_n$ of X which is a k_ω -space and hence Hausdorff, whence x and y can be separated by open neighbourhoods in U . \square

Remark 4.6 Hausdorff quotients of k_ω -spaces being k_ω , it easily follows from Proposition 4.5 that the direct limit Hausdorff topological space of each countable direct system of k_ω -spaces is a k_ω -space.

To formulate a result concerning the compatibility of direct limits and direct products, let $X_1 \subseteq X_2 \subseteq \dots$ and $Y_1 \subseteq Y_2 \subseteq \dots$ be ascending sequences of topological spaces with continuous inclusion maps, and $X := \bigcup_{n \in \mathbb{N}} X_n = \varinjlim X_n$, $Y := \bigcup_{n \in \mathbb{N}} Y_n$, equipped with the direct limit topology. Write $\varinjlim (X_n \times Y_n)$ for $\bigcup_{n \in \mathbb{N}} (X_n \times Y_n)$, equipped with the direct limit topology.

Proposition 4.7 *The natural map*

$$\beta: \varinjlim (X_n \times Y_n) \rightarrow (\varinjlim X_n) \times (\varinjlim Y_n), \quad (x, y) \mapsto (x, y)$$

is a continuous bijection. If each X_n and each Y_n is locally k_ω , then β is a homeomorphism.

Proof. The inclusion map $X_n \times Y_n \rightarrow X \times Y$ is continuous for each $n \in \mathbb{N}$. Therefore β is continuous. If each X_n and Y_n is locally k_ω , let $(x, y) \in X \times Y$, say $(x, y) \in X_1 \times Y_1$. We show that there exist open neighbourhoods $U \subseteq X$ of x and $V \subseteq Y$ of y such that $X \times Y$ and $\varinjlim (X_n \times Y_n)$ induce the same topology on $U \times V$. To this end, let U_1 be an

open neighbourhood of x in X_1 which is k_ω , and choose open subsets $U_n \subseteq X_n$ such that $U_n \subseteq U_{n+1}$ for each n and each U_n is k_ω (using Lemma 4.3). Likewise, choose open, k_ω subsets $V_n \subseteq Y_n$ such that $y \in V_1$ and $V_n \subseteq V_{n+1}$. Then $U := \bigcup_{n \in \mathbb{N}} U_n$ and $V := \bigcup_{n \in \mathbb{N}} V_n$ are k_ω -spaces in the topology induced by X and Y , respectively (as a consequence of Proposition 4.2 (d) and (e)). As in the first half of the proof of Proposition 4.5, we find k_ω -sequences $(K_n^{(j)})_{n \in \mathbb{N}}$ for each U_j such that $K_n^{(i)} \subseteq K_n^{(j)}$ for $i \leq j$, whence $K_n := K_n^{(n)}$ defines a k_ω -sequence for U . Choose k_ω -sequences $(L_n^{(j)})_{n \in \mathbb{N}}$ for V_j analogously and define $L_n := L_n^{(n)}$. By the compatibility of direct products and countable direct limits of compact spaces (see, e.g., [27, Proposition 3.3]), we have

$$U_j \times V_j = \left(\varinjlim K_n^{(j)} \right) \times \left(\varinjlim L_n^{(j)} \right) = \varinjlim (K_n^{(j)} \times L_n^{(j)}),$$

showing that $(K_n^{(j)} \times L_n^{(j)})_{n \in \mathbb{N}}$ is a k_ω -sequence for $U_j \times V_j$. Hence $(K_n \times L_n)_{n \in \mathbb{N}}$ is a k_ω -sequence for the open subset $\varinjlim (U_n \times V_n)$ of $\varinjlim (X_n \times Y_n)$, equipped with the induced topology. Then

$$U \times V = \left(\varinjlim K_n \right) \times \left(\varinjlim L_n \right) = \varinjlim (K_n \times L_n) = \varinjlim (U_n \times V_n),$$

using [27, Proposition 3.3] for the second equality. Here $\varinjlim (U_n \times V_n)$ is the set $U \times V$, equipped with the topology induced by $\varinjlim (X_n \times Y_n)$ (see Lemma 1.1 (a)). Thus $X \times Y$ and $\varinjlim (X_n \times Y_n)$ induce the same topology on this set, which completes the proof. \square

With a view towards later applications to Kac-Moody groups, we recall a simple fact from [24, § 21]. It illustrates that metrizability is not preserved under passage to direct limits.

Proposition 4.8 *If a locally k_ω space X is metrizable, then it is locally compact.* \square

5 Locally k_ω groups and their direct limits

In this section, we introduce and discuss the notion of a locally k_ω group. We also prove the existence of direct limits for ascending sequences of locally k_ω groups, and related results.

Definition 5.1 A k_ω -group (resp., locally k_ω group) is a topological group the underlying topological space of which is a k_ω -space (resp., a space which is locally k_ω).

Example 5.2 Among the main examples of infinite-dimensional Lie groups are direct limits $G = \varinjlim G_n$ of ascending sequences $G_1 \subseteq G_2 \subseteq \dots$ of finite-dimensional Lie groups, such that the inclusion maps are continuous homomorphisms (see [27], [28], [70]). As G is equipped with the final topology here, Proposition 4.5 shows that the topological space underlying G is locally k_ω . Thus G is a locally k_ω group. The same conclusion holds for countable direct limits of locally compact groups, as considered in [27] and [82]. If each G_n is σ -compact, then G is a k_ω -group. However, if some G_n fails to be σ -compact and each inclusion map is a topological embedding, then G is not a k_ω -group.

Proposition 5.3 *For a topological group G , the following conditions are equivalent:*

- (a) G is a locally k_ω group;
- (b) G has an open subgroup $H \leq G$ which is a k_ω -group;
- (c) $G = \varinjlim X_n$ as a topological space for an ascending sequence $X_1 \subseteq X_2 \subseteq \cdots$ of closed, locally compact subsets X_n of G , equipped with the induced topology.

Proof. (a) \Rightarrow (b): Let U_1 be an open identity neighbourhood of G which is a k_ω -space. Then $U_1 U_1 \cup U_1^{-1}$ is a σ -compact subset of G and hence has an open neighbourhood U_2 which is a k_ω -space. Proceeding in this way, we obtain an ascending sequence $U_1 \subseteq U_2 \subseteq \cdots$ of open identity neighbourhoods U_n which are k_ω -spaces, and such that $U_n U_n \cup U_n^{-1} \subseteq U_{n+1}$ for each $n \in \mathbb{N}$. Then $U := \bigcup_{n \in \mathbb{N}} U_n$ is a subgroup of G . Apparently, U is open in G and $U = \varinjlim U_n$, which is a k_ω -space by Proposition 4.5.

(b) \Rightarrow (c): Let $H \subseteq G$ be an open subgroup which is a k_ω -group, and $(K_n)_{n \in \mathbb{N}}$ be a k_ω -sequence for H . Let $T \subseteq G$ be a transversal for G/H . Then $X_n := \bigcup_{g \in T} gK_n$ is a closed subset of G , since $(gK_n)_{g \in T}$ is a locally finite family of closed sets (see [79, §8.5, Hilfssatz 2]). Furthermore, X_n is locally compact, because $gK_n = X_n \cap gH$ is open in X_n . Then $G = \bigcup_{n \in \mathbb{N}} X_n$ and $X_1 \subseteq X_2 \subseteq \cdots$. The final topology \mathcal{T} on G with respect to the inclusion maps $X_n \rightarrow G$ is finer than the original topology \mathcal{O} . If $U \in \mathcal{T}$, let $g \in T$. Then $(U \cap gH) \cap gK_n = (U \cap X_n) \cap gK_n$ is open in gK_n for each $n \in \mathbb{N}$. Since $(gK_n)_{n \in \mathbb{N}}$ is a k_ω -sequence for gH , this entails that $U \cap gH$ is open in gH and hence open in G . Hence $U = \bigcup_{g \in T} (U \cap gH) \in \mathcal{O}$. Thus $\mathcal{O} = \mathcal{T}$.

(c) \Rightarrow (a): Locally compact spaces being locally k_ω , (c) entails that G is a direct limit of an ascending sequence of locally k_ω spaces and hence locally k_ω , by Proposition 4.5. \square

Proposition 5.4 *Let $G_1 \subseteq G_2 \subseteq \cdots$ be an ascending sequence of k_ω -groups (resp., locally k_ω groups) such that the inclusion maps $G_n \rightarrow G_{n+1}$ are continuous homomorphisms. Equip $G := \bigcup_{n \in \mathbb{N}} G_n$ with the unique group structure making each inclusion map $G_n \rightarrow G$ a homomorphism. Then the final topology with respect to the inclusion maps $G_n \rightarrow G$ turns G into a k_ω -group (resp., a locally k_ω group) and makes G the direct limit $\varinjlim G_n$ in the category of topological spaces, and in the category of topological groups.*

Proof. The direct limit property in the category of topological spaces is clear, and we know from Proposition 4.5 that G is a k_ω -space (resp., a locally k_ω space). Let $\mu_n: G_n \times G_n \rightarrow G_n$ and $\lambda_n: G_n \rightarrow G_n$ be the group multiplication (resp., inversion) of G_n , and $\mu: G \times G \rightarrow G$ and $\lambda: G \rightarrow G$ be the group multiplication (resp., inversion) of G . Then $\lambda = \varinjlim \lambda_n$, where each λ_n is continuous, and hence λ is continuous. Identifying $G \times G$ with $\varinjlim (G_n \times G_n)$ as a topological space by means of Proposition 4.7, the group multiplication μ becomes the map $\varinjlim \mu_n: \varinjlim (G_n \times G_n) \rightarrow \varinjlim G_n$, which is continuous. Hence G is a topological group. The remainder is clear. \square

Corollary 5.5 *Let I be a countable directed set and $\mathcal{S} := ((G_i)_{i \in I}, (f_{ij})_{i \geq j})$ be a direct system of k_ω -groups (resp., locally k_ω groups) and continuous homomorphisms. Then \mathcal{S} has a direct limit $(G, (f_i)_{i \in I})$ in the category of Hausdorff topological groups. The topology on G is the final topology with respect to the family $(f_i)_{i \in I}$, and the Hausdorff topological space underlying G is the direct limit of \mathcal{S} in the category of Hausdorff topological spaces. Furthermore, G is a k_ω -group (resp., locally k_ω group) and hence is the direct limit of \mathcal{S} in the category of k_ω -groups (resp., locally k_ω groups).*

Proof. We may assume that $I = (\mathbb{N}, \leq)$. Let $N_m := \overline{\bigcup_{n \geq m} \ker f_{n,m}}$, $Q_m := G_m/N_m$ and $q_m: G_m \rightarrow Q_m$ be the canonical quotient morphism. By Proposition 4.2(d) (resp., (j)), each Q_m is a k_ω -group (resp., a locally k_ω group). Let $g_{n,m}: Q_m \rightarrow Q_n$ be the continuous homomorphism determined by $g_{n,m} \circ q_m = q_n \circ f_{n,m}$. Assume that $(X, (h_n)_{n \in \mathbb{N}})$ is a cone of continuous maps $h_n: G_n \rightarrow X$ to a Hausdorff topological space. Then each h_n factors to a continuous map $k_n: Q_n \rightarrow X$. From the preceding, it is clear that the direct limit of \mathcal{S} in the category of Hausdorff topological spaces (resp., Hausdorff topological groups) coincides with that of $((Q_n)_{n \in \mathbb{N}}, (g_{n,m})_{n \geq m})$. As each $g_{n,m}$ is injective, the assertions follow from Proposition 5.4. \square

Remark 5.6 If each of the groups G_i in Proposition 5.5 is abelian, then also G is abelian, whence $G = \varinjlim G_i$ in the category of abelian k_ω -groups (resp., abelian, locally k_ω groups). Thus [4, Prop. 2.1] is obtained as special case, the proof of which given in [4] is incorrect.³

Inspired by arguments from the proof of [4, Proposition 2.1], we record:

Corollary 5.7 *Let $(G_n)_{n \in \mathbb{N}}$ be a sequence of abelian k_ω -groups (resp., abelian, locally k_ω groups). Then the box topology on $G := \bigoplus_{n \in \mathbb{N}} G_n$ makes G the direct limit $\varinjlim \prod_{k=1}^n G_k$ in the category of topological spaces, and this topology is k_ω (resp., locally k_ω).*

Proof. By Proposition 4.2(c) (resp., (i)), $\prod_{k=1}^n G_k$ is a k_ω -group (resp., locally k_ω). By Proposition 5.4, the direct limit topology turns $\bigoplus_{n \in \mathbb{N}} G_n = \varinjlim \prod_{k=1}^n G_k$ into a k_ω -group (resp., locally k_ω group), and it makes $\bigoplus_{n \in \mathbb{N}} G_n$ the direct limit $\varinjlim \prod_{k=1}^n G_k$ in the category \mathbb{TG} of topological groups. But it is well known that the topology making $\bigoplus_{n \in \mathbb{N}} G_n$ the direct limit $\varinjlim \prod_{k=1}^n G_k$ in \mathbb{TG} is the box topology (see, e.g., [29, Lemma 4.4]). \square

Cf. also [19] for recent investigations of topologies on direct sums.

The following proposition is a variant of [63, Theorem 1], which corresponds to the most essential case where the final group topology with respect to a single map $f: X \rightarrow G$ from a k_ω -space X to G is Hausdorff and makes f a topological embedding (cf. also [72], which even subsumes non-Hausdorff situations).

³The problem is that the final group topology on $S := \bigoplus_{n \in \mathbb{N}} G_n$ with respect to the inclusion maps $i_n: G_n \rightarrow S$ does not coincide with the final topology with respect to these maps.

Proposition 5.8 *Let G be a group and $(f_i)_{i \in I}$ be a countable family of maps $f_i: X_i \rightarrow G$ such that each X_i is a k_ω -space and $\bigcup_{i \in I} f_i(X_i)$ generates G . If the final group topology on G with respect to the family $(f_i)_{i \in I}$ is Hausdorff, then it makes G a k_ω -group.*

Proof. The disjoint union $X := \coprod_{i \in I} X_i$ is a k_ω -space (by Proposition 4.2 (e)) and hence a regular topological space (see [41, Proposition 4.3 (ii)]). We can therefore form the Markov free topological group F on X , as in [64]. The final group topology on G turns G into a Hausdorff quotient group of F . Since X is a k_ω -space, F is a k_ω -group [63, Corollary 1 (a)]. Hence also its Hausdorff quotient group G is a k_ω -group, by Proposition 4.2 (d). \square

Remark 5.9 Note that, in the situation of Proposition 5.8, the final group topology on G is Hausdorff if there exists a Hausdorff group topology on G making all maps f_i continuous.

We record a direct consequence concerning colimits in the category \mathbb{KOG} of k_ω -groups.

Corollary 5.10 *Let $\delta: \mathbb{I} \rightarrow \mathbb{KOG}$ be a diagram of k_ω -groups $G_i := \delta(i)$ for $i \in I := \text{ob}(\mathbb{I})$ and continuous homomorphisms $\varphi_\alpha := \delta(\alpha): G_i \rightarrow G_j$ for $i, j \in I$ and $\alpha \in \text{Mor}(i, j)$, such that I is countable. Let $(G, (\lambda_i)_{i \in I})$ be a colimit of δ in the category of Hausdorff topological groups, with homomorphisms $\lambda_i: G_i \rightarrow G$. Then G is a k_ω -group, and $(G, (\lambda_i)_{i \in I})$ also is the colimit of δ in the category of k_ω -groups.* \square

Lemma 5.11 *Quotient morphisms between locally k_ω groups are compact-covering.*

Proof. Given a quotient morphism $q: G \rightarrow Q$, let $U \subseteq G$ be an open subgroup which is a k_ω -group. Then $q(U)$ is an open subgroup of Q and k_ω , by Lemma 4.2 (d). Given a compact set $K \subseteq Q$, there are $y_1, \dots, y_n \in K$ such that $K \subseteq \bigcup_{j=1}^n y_j q(U)$. By Lemma 4.4, $y_j^{-1}(K \cap y_j q(U)) \subseteq q(L_j)$ for a compact set $L_j \subseteq U$ and thus $K \subseteq q(L)$ with $L := \bigcup_{j=1}^n x_j L_j$, where $x_j \in q^{-1}(\{y_j\})$. \square

Lemma 5.12 *In the situation of Corollary 5.5, let $K \subseteq G$ be a compact subset. Then $K \subseteq f_i(L)$ for some $i \in I$ and some compact subset $L \subseteq G_i$.*

Proof. Since the quotient morphisms $q_i: G_i \rightarrow Q_i$ in the proof of Corollary 5.5 are compact-covering by Lemma 5.11, we may assume that each f_{ij} (and hence each f_i) is injective. It therefore suffices to consider the situation of Proposition 5.4. Using Lemma 4.3 and the argument from the proof of “(a) \Rightarrow (b)” in Proposition 5.3, for each $n \in \mathbb{N}$ we find an open subgroup $U_n \subseteq G_n$ which is a k_ω -group, such that $U_n \subseteq U_{n+1}$ for each $n \in \mathbb{N}$. Then $U := \bigcup_{n \in \mathbb{N}} U_n$ is an open subgroup of $G = \bigcup_{n \in \mathbb{N}} G_n$ (see Lemma 1.1 (a)), and the induced topology makes U the direct limit $U = \varinjlim U_n$. By Proposition 4.5, U is a k_ω -space. Choose a k_ω -sequence $(K_n^{(j)})_{n \in \mathbb{N}}$ for U_j , for each $j \in \mathbb{N}$, such that $K_n^{(i)} \subseteq K_n^{(j)}$ whenever $i \leq j$. Then $(K_n^{(n)})_{n \in \mathbb{N}}$ is a k_ω -sequence for U . Given a compact subset $K \subseteq U$, we have $K \subseteq K_n^{(n)}$ for some $n \in \mathbb{N}$ (Lemma 1.1 (d)), whence K is a compact subset of U_n . Since every compact subset of G is contained in finitely many translates of U , the assertion easily transfers from compact subsets of U to those of G (cf. the preceding proof). \square

6 Complex Kac-Moody groups and their unitary forms

In this section, we generalize our results from Section 3 about compact Lie groups — i.e. unitary forms of complex Lie groups with respect to the compact involution — to unitary forms of complex Kac-Moody groups with respect to the compact involution. These groups correspond to complex Kac-Moody algebras in very much the same way that finite-dimensional semisimple complex Lie groups correspond to finite-dimensional semisimple complex Lie algebras. As far as complex Kac-Moody algebras are concerned, we use the notations from [49]: If A is a generalized Cartan matrix then $\mathfrak{g}(A)$ is the associated Kac-Moody Lie algebra and $\mathfrak{g}'(A)$ is its derived Lie algebra. The root space decomposition $\mathfrak{g}(A) = \mathfrak{h} \oplus \bigoplus \mathfrak{g}_\alpha$ induces a decomposition $\mathfrak{g}'(A) = \mathfrak{h}' \oplus \bigoplus \mathfrak{g}_\alpha$ and by *loc.cit.*, Chapter 5, the set of roots Δ decomposes into the set Δ^{re} of real roots and the set Δ^{im} of imaginary roots. If W denotes the Weyl group of $\mathfrak{g}(A)$ then the real roots are canonically identified with the roots of a certain Coxeter system (W, S) defined as in *loc.cit.*, 3.13. For any real root α , the root space \mathfrak{g}_α is integrable.

Convention: We assume throughout that the Coxeter system (W, S) associated with A is two-spherical and of finite rank $N := |S|$.

We now compile definitions, notation and facts concerning complex Kac-Moody groups which will be used in the sequel. For proofs, see [78] and the references therein. By definition, the *complex Kac-Moody group* $G(A)$ is the group associated to the (integrable representations of the) complex Lie algebra $\mathfrak{g}'(A)$ in the sense of [48], where it is also shown that this definition is equivalent to the one given in [50]. (For a more algebraic construction of Kac-Moody groups which works over arbitrary fields, see [86].) In particular, if $F_{\mathfrak{g}'(A)}$ denotes the set of ad-locally finite elements in $\mathfrak{g}'(A)$, then there exists a partial exponential function $\exp: F_{\mathfrak{g}'(A)} \rightarrow G(A)$, which is uniquely determined by the property that for any integrable $\mathfrak{g}'(A)$ -module $(V, d\pi)$, there exists a representation π of $G(A)$ on V such that for any $v \in V$ the equality $(\pi \circ \exp(X)).v = \sum_{n=0}^{\infty} \frac{(d\pi(X))^n.v}{n!}$ holds. For any real root $\alpha \in \Delta^{re}$, we define the associated *root group* to be $U_\alpha := \exp \mathfrak{g}_\alpha$. If $H := \bigcap_{\alpha \in \Delta^{re}} N_{G(A)}(U_\alpha)$, then the triple $(G, (U_\alpha)_{\alpha \in \Delta^{re}}, H)$ forms a root group data (RGD) system (*donnée radicielle jumelée*) of type (W, S) in the sense of [78, 1.5]. In particular, if $U_\pm := \langle U_\alpha \mid \alpha \in \Delta_\pm^{re} \rangle$, then $B_\pm := HU_\pm$ are opposite Borel groups in $G(A)$ and one has the Bruhat and Birkhoff decompositions

$$G(A) = \bigcup_{w \in W} B^+ w B^+ = \bigcup_{w \in W} B^- w B^- = \bigcup_{w \in W} B^+ w B^-.$$

For a real root α the Lie algebra $\mathfrak{g}(\alpha)$ generated by \mathfrak{g}_α and $\mathfrak{g}_{-\alpha}$ in $\mathfrak{g}'(A)$ is isomorphic to \mathfrak{sl}_2 and contained in $F_{\mathfrak{g}'(A)}$. Therefore the inclusion map $\varphi_\alpha: \mathfrak{sl}_2 \cong \mathfrak{g}(\alpha) \hookrightarrow \mathfrak{g}'(A)$ is integrable in the sense of [48] and thus gives rise to a map $\varphi_\alpha: \mathrm{SL}_2(\mathbb{C}) \rightarrow G(A)$, whose image is denoted G_α . It is known that $G_\alpha = \langle U_\alpha, U_{-\alpha} \rangle$ and that φ_α is an isomorphism onto its image. (See [50] for details.) We call G_α the *rank one subgroup* associated with the real root α . If $\alpha_1, \dots, \alpha_n$ are real roots, then we define the associated *rank n subgroup* to be

$G_{\alpha_1, \dots, \alpha_n} := \langle G_{\alpha_1}, \dots, G_{\alpha_n} \rangle$. The rank one subgroups can be used to define a topology on $G(A)$ as was first observed by Kac and Peterson in [50]:

Definition 6.1 The final group topology with respect to the inclusion maps $\varphi_\alpha: \mathrm{SL}_2(\mathbb{C}) \rightarrow G(A)$, where α runs through the real roots of $\mathfrak{g}(A)$ and $\mathrm{SL}_2(\mathbb{C})$ is equipped with its connected Lie group topology, is called the *Kac-Peterson topology* on $G(A)$ and denoted τ_{KP} .

We record an easy fact for later reference:

Lemma 6.2 *If $\Pi = \{\alpha_1, \dots, \alpha_N\}$ is any choice of simple roots for $\mathfrak{g}(A)$, then τ_{KP} is the final group topology with respect to the inclusions $\varphi_{\alpha_i}: \mathrm{SL}_2(\mathbb{C}) \rightarrow G(A)$, $i = 1, \dots, N$.*

Proof. The Weyl group W can be identified with $N_{G(A)}(H)/H$ and as H normalizes every root group, W acts by conjugation on root groups. This action is equivariant with respect to the action on real roots, i.e. for any real root α and every $w \in W$ one has $wU_\alpha w^{-1} = U_{w.\alpha}$ and thus $wG_\alpha w^{-1} = G_{w.\alpha}$. Now by definition every real root α of $\mathfrak{g}'(A)$ is of the form $w.\alpha_i$ for some $\alpha_i \in \Pi$ and thus every rank one subgroup is conjugate to some G_{α_j} . Hence if τ is a group topology on $G(A)$ for which each of the inclusions $\varphi_{\alpha_i}: \mathrm{SL}_2(\mathbb{C}) \rightarrow G(A)$ is continuous for $i = 1, \dots, N$, then by continuity of conjugation all rank one subgroup inclusions are in fact continuous. This implies the lemma. \square

The following theorem relates complex Kac-Moody groups to the main theme of this article:

Theorem 6.3 *The topological group $(G(A), \tau_{KP})$ is a k_ω -group.*

To prove Theorem 6.3, we need some preliminary results on integrable representations.

Lemma 6.4 *Let $(V, d\psi)$ be an integrable $\mathfrak{g}'(A)$ -module and $\psi: G(A) \rightarrow \mathrm{Aut}(V)$ be the associated representation of $G(A)$. Let α be a real root. Then every $v \in V$ is contained in a finite-dimensional G_α -submodule V_0 of V , and the orbit map $G_\alpha \rightarrow V$, $g \mapsto \psi(g).v$ is continuous into V_0 . Moreover, for any $v \in V$ and real roots $\alpha_1, \dots, \alpha_n$, the map*

$$G_{\alpha_1} \times \dots \times G_{\alpha_n} \rightarrow V, \quad (g_1, \dots, g_n) \mapsto \psi(g_1 \dots g_n).v \quad (2)$$

has image in a finite-dimensional vector subspace of V and is continuous into this space.

Proof. Since V is integrable and $\mathfrak{g}_\alpha \subseteq F_{\mathfrak{g}'(A)}$ is finite-dimensional, v is contained in a finite-dimensional \mathfrak{g}_α -submodule V_0 of V (see [49, Proposition 3.6 (a)] or part (b) of the lemma in [48, §1.2]). Then $\psi(\exp(X)).w = \sum_{n=0}^{\infty} \frac{(d\psi(X))^n.w}{n!} \in V_0$ for all $X \in \mathfrak{g}_\alpha$ and $w \in V_0$, entailing that V_0 is a G_α -submodule and that $\pi: G_\alpha \rightarrow \mathrm{GL}(V_0)$, $g \mapsto \psi(g)|_{V_0}$ is the smooth (and hence continuous) representation of the Lie group G_α with $d\pi(X) = d\psi(X)$ for $X \in \mathfrak{g}_\alpha$. As a consequence, the orbit map $G_\alpha \rightarrow V_0$, $g \mapsto \psi(g).v = \pi(g).v$ is continuous.

To prove the final assertion, we proceed by induction. The case $n = 1$ has just been settled. As the inductive hypothesis, we may now assume that $\psi(G_{\alpha_2} G_{\alpha_3} \dots G_{\alpha_n}).v$ is contained in a finite-dimensional vector subspace $V_0 \subseteq V$ and that the map $\varphi: G_{\alpha_2} \times \dots \times G_{\alpha_n} \rightarrow V_0$,

$\varphi(g_2, \dots, g_n) := \psi(g_2 \cdots g_n).v$ is continuous. Each element of V being contained in a finite-dimensional \mathfrak{g}_{α_1} -module, the \mathfrak{g}_{α_1} -module $V_1 \subseteq V$ generated by V_0 is finite-dimensional. Now consider the map $\pi: G_{\alpha_1} \rightarrow \mathrm{GL}(V_1)$, $g \mapsto \psi(g)|_{V_1}$. By the case $n = 1$, the map $G_{\alpha_1} \rightarrow V_1$, $g \mapsto \pi(g).w$ is continuous for each $w \in V_1$, entailing that π is continuous. The evaluation map $\varepsilon: \mathrm{GL}(V_1) \times V_1 \rightarrow V_1$, $(B, x) \mapsto B(x)$ being continuous, it follows that $\varepsilon \circ (\pi \times \varphi): G_{\alpha_1} \times (G_{\alpha_2} \times \cdots \times G_{\alpha_n}) \rightarrow V_1$ is continuous. This is the mapping in (2). \square

Now we can deduce Theorem 6.3 as follows:

Proof of Theorem 6.3. Fix a system of simple roots $\Pi = \{\alpha_1, \dots, \alpha_N\}$ and for every word $I := (i_1, \dots, i_k)$ over $\{1, \dots, N\}$ denote by G_I the image of the map

$$p_I: G_{\alpha_{i_1}} \times \cdots \times G_{\alpha_{i_k}} \rightarrow G(A), \quad (g_1, \dots, g_k) \mapsto g_1 \cdots g_k,$$

which we equip with the quotient topology τ_I with respect to the map p_I , where G_{α_j} is identified with $\mathrm{SL}_2(\mathbb{C})$. We claim that G_I is a k_ω -space for every I . As $\mathrm{SL}_2(\mathbb{C})^k$ is a k_ω -space, it suffices to show (by Part (d) of Proposition 4.2) that G_I is Hausdorff. For this, let $g \neq h \in G_I$. By definition of $G(A)$, there exists an integrable $\mathfrak{g}'(A)$ -module $(V, d\psi)$ such that $\psi(g) \neq \psi(h)$, where $\psi: G(A) \rightarrow \mathrm{Aut}(V)$ is the representation associated to $d\psi$. Thus there is $v \in V$ such that $\psi(g).v \neq \psi(h).v$. Define $f: G_I \rightarrow V$, $f(x) := \psi(x).v$. The map $f \circ p_I$ is continuous by Lemma 6.4. Hence also f is continuous. Now suitable preimages of f provide disjoint neighbourhoods of g and h , finishing the proof that G_I is Hausdorff and k_ω . On the set \mathcal{W}_N of words over $\{1, \dots, N\}$, define a partial ordering by putting $I \leq J$ if and only if I is a subsequence of J . Then for $I \leq J$ there is an obvious inclusion map $G_I \rightarrow G_J$. Let τ_0 denote the final topology with respect to the system $(G_I, \tau_I)_{I \in \mathcal{W}_N}$ on the union $G(A) = \bigcup G_I$. A cofinal subsequence $I_1 \leq I_2 \leq \cdots$ in \mathcal{W}_N can be defined by $I_1 := (1)$, $I_2 := (1, 2)$, \dots , $I_{N+1} := (1, 2, \dots, N, 1)$ and so on. We then have

$$G_{I_1} \subseteq G_{I_2} \subseteq \cdots \subseteq G(A)$$

and as each of the spaces G_{I_n} is k_ω , we conclude from Proposition 4.5 that $(G(A), \tau_0)$ is a k_ω -space. It is easy to check that τ_0 is a group topology: The concatenation map $G_{(i_1, \dots, i_k)} \times G_{(j_1, \dots, j_m)} \rightarrow G_{(i_1, \dots, i_k, j_1, \dots, j_m)}$ is continuous and thus $G_{(i_1, \dots, i_k)} \times G_{(j_1, \dots, j_m)} \rightarrow G(A)$ is continuous. Then by Proposition 4.7, multiplication is continuous and the proof for continuity of inversion is similar. This shows that $(G(A), \tau_0)$ is a k_ω -group. The inclusion maps $\varphi_{\alpha_i}: \mathrm{SL}_2(\mathbb{C}) \rightarrow (G(A), \tau_0)$, $i = 1, \dots, N$ are continuous by construction. On the other hand, if τ_1 is any group topology on $G(A)$ for which these inclusions are continuous, then also the maps $p_I: G_{\alpha_{i_1}} \times \cdots \times G_{\alpha_{i_k}} \rightarrow G(A)$ are continuous. In view of Lemma 6.2, this implies the theorem. \square

We record a byproduct of the proof of Theorem 6.3.

Corollary 6.5 *Let $(V, d\psi)$ be an integrable $\mathfrak{g}'(A)$ -module and $\psi: G(A) \rightarrow \mathrm{Aut}(V)$ be the associated representation of $G(A)$. Then ψ is continuous with respect to the Kac-Peterson topology on $G(A)$ and the topology of pointwise convergence on $\mathrm{Aut}(V)$.*

Proof. For each of the sets G_I , $I \in \mathcal{W}_N$ as defined in the proof of Theorem 6.3, the restriction $G_I \rightarrow \text{Aut}(V)$ of ψ is continuous by Lemma 6.4. As $G(A) = \varinjlim G_I$, the assertion follows. \square

Let us call a subalgebra \mathfrak{h} of $\mathfrak{g}'(A)$ *integrable* if every $\text{ad}_{\mathfrak{h}}$ -locally finite element of \mathfrak{h} is $\text{ad}_{\mathfrak{g}'(A)}$ -locally finite. (For a complex subalgebra \mathfrak{h} this is equivalent to the condition that the inclusion map $\mathfrak{h} \rightarrow \mathfrak{g}'(A)$ is integrable in the sense of [48].) As usual, a subalgebra $\mathfrak{g}^0(A) \subset \mathfrak{g}'(A)$ is called a *real form* if $\mathfrak{g}'(A) = \mathfrak{g}^0(A) \oplus i\mathfrak{g}^0(A) \cong \mathfrak{g}^0(A) \otimes \mathbb{C}$. A real form $\mathfrak{g}^0(A)$ induces an involutive Lie algebra automorphism σ on $\mathfrak{g}'(A)$ by

$$\sigma(X_0 + iX_1) := X_0 - iX_1 \quad (X_0, X_1 \in \mathfrak{g}^0(A)).$$

We observe:

Proposition 6.6 *Let $\mathfrak{g}^0(A) \subset \mathfrak{g}'(A)$ be a real form and $\sigma: \mathfrak{g}'(A) \rightarrow \mathfrak{g}'(A)$ the associated involution. Then the following hold:*

- (a) $\mathfrak{g}^0(A) \subseteq \mathfrak{g}'(A)$ is an integrable subalgebra.
- (b) σ is integrable and the induced involution $\sigma: G(A) \rightarrow G(A)$ on the group level is continuous.

Proof. (a) Let us denote by ad and ad_0 the adjoint representations of $\mathfrak{g}'(A)$ and $\mathfrak{g}^0(A)$, respectively. Let $X \in \mathfrak{g}^0(A)$ be ad_0 -locally finite and let $Y \in \mathfrak{g}'(A)$. We may write $Y = Y_0 + iY_1$ with $Y_0, Y_1 \in \mathfrak{g}^0(A)$. By assumption, there exist finite-dimensional $\text{ad}_0(X)$ -invariant subspaces V_0 and V_1 of $\mathfrak{g}^0(A)$ containing Y_0 and Y_1 , respectively. We thus have $Y \in V_0 \oplus iV_1$ and for $Z = Z_0 + iZ_1 \in V_0 \oplus iV_1$, we obtain

$$\text{ad}(X)(Z) = [X, Z_0 + iZ_1] = [X, Z_0] + i[X, Z_1] = \text{ad}_0(X)(Z_0) + i\text{ad}_0(X)(Z_1) \in V_0 \oplus iV_1,$$

showing that $V_0 \oplus iV_1$ is a finite-dimensional $\text{ad}(X)$ -invariant subspace containing Y . As $Y \in \mathfrak{g}'(A)$ was arbitrary, we conclude that X is ad -locally finite, which proves (a).

(b) Suppose that X is ad -locally finite. For $Y \in \mathfrak{g}'(A)$, let V be a finite-dimensional $\text{ad}(X)$ -invariant subspace of $\mathfrak{g}'(A)$ containing $\sigma(Y)$. Put $V' := \sigma(V)$. We claim that V' is $\text{ad}(\sigma(X))$ -invariant. Indeed, let $Z' \in V'$ with $Z' = \sigma(Z)$ for some $Z \in V$. Then

$$\text{ad}(\sigma(X))(Z') = [\sigma(X), \sigma(Z)] = \sigma([X, Z]) = \sigma(\text{ad}(X)(Z)) \in \sigma(V) = V'.$$

Moreover, we have $Y = \sigma(\sigma(Y)) \in V'$ and thus V' is a finite-dimensional $\text{ad}(\sigma(X))$ -invariant subspace containing Y . As Y was arbitrary, $\sigma(X)$ is ad -locally finite and thus σ is integrable. The induced involution $\sigma: G(A) \rightarrow G(A)$ on $G(A)$ is determined by the fact that $\sigma(\exp(X)) = \exp(\sigma(X))$. This formula shows at once that the restriction of σ to any rank one subgroup is smooth, which implies that σ is continuous. \square

If $\mathfrak{g}^0(A) \subseteq \mathfrak{g}'(A)$ is a real form with associated involution σ , then the group $G^0(A) := G(A)^\sigma$ is called the *real form* of $G(A)$ associated with $\mathfrak{g}^0(A)$. We equip $G^0(A)$ with the topology induced by $G(A)$.

Corollary 6.7 *Every real form of a complex Kac-Moody group is a k_ω -group.*

Proof. Combine Proposition 6.6 (b) with Proposition 4.2 (b) and Theorem 6.3. \square

In view of Part (a) of Proposition 6.6, one can think of another approach towards real forms of complex Kac-Moody groups: The general construction from [48] associates to (the integrable complex representations of) any complex Lie algebra \mathfrak{h} an associated group H . The same construction can also be applied to the integrable representations of any real Lie algebra. In particular, we can use it to associate a group G^0 to any given real form $\mathfrak{g}^0(A) \subseteq \mathfrak{g}'(A)$. In view of Part (a) of Proposition 6.6, the inclusion $\mathfrak{g}^0(A) \rightarrow \mathfrak{g}'(A)$ induces a map $i: G^0 \rightarrow G(A)$. Then one has:

Proposition 6.8 *The inclusion $\mathfrak{g}^0(A) \rightarrow \mathfrak{g}'(A)$ induces an injection $i: G^0 \rightarrow G^0(A) \subseteq G(A)$, where $G^0(A)$ is the real form of $G(A)$ associated with $\mathfrak{g}^0(A)$.*

Proof. Let $g \in G^0$. If $g \neq e$, there exists an integrable representation $d\varphi: \mathfrak{g}^0(A) \rightarrow \text{End}(V^0)$ with associated group representation $\varphi: G^0(A) \rightarrow \text{Aut}(V^0)$ such that $\varphi(g) \neq \text{Id}_{V^0} = \varphi(e)$. We can complexify $d\varphi$ to a Lie algebra homomorphism $d\varphi_{\mathbb{C}}: \mathfrak{g}^0(A) \otimes \mathbb{C} \rightarrow \text{End}(V^0 \otimes \mathbb{C})$ by $d\varphi_{\mathbb{C}}(X \otimes z)(v) := d\varphi(X)(zv)$, which we may identify with a Lie algebra homomorphism $d\varphi_{\mathbb{C}}: \mathfrak{g}'(A) \rightarrow \text{End}(V^0 \otimes \mathbb{C})$ by identifying $\mathfrak{g}^0(A) \otimes \mathbb{C}$ with $\mathfrak{g}'(A)$. It is easy to see that $d\varphi_{\mathbb{C}}$ is also integrable. We thus obtain a commuting square

$$\begin{array}{ccc} G(A) & \xrightarrow{\varphi_{\mathbb{C}}} & \text{Aut}(V^0 \otimes \mathbb{C}), \\ \uparrow i & & \uparrow \\ G^0(A) & \xrightarrow{\varphi} & \text{Aut}(V^0) \end{array}$$

where $\text{Aut}(V) \rightarrow \text{Aut}(V \otimes \mathbb{C})$ is induced from the inclusion $V \rightarrow V \otimes \mathbb{C}$, $v \mapsto v \otimes 1$. As $\varphi(g) \neq \text{Id}_{V^0}$, we have also $\varphi_{\mathbb{C}}(i(g)) \neq \text{Id}_{V^0 \otimes \mathbb{C}} = \varphi_{\mathbb{C}}(i(e))$ and thus $i(g) \neq i(e)$, showing that i is injective. Since G^0 is generated by the elements $\exp(X)$ for $X \in \mathfrak{g}^0(A)$, which are σ -invariant, we see that $i(G^0) \subseteq G^0(A)$. \square

Proposition 6.8 suggests the question of whether the map i actually defines an isomorphism $G^0 \cong G^0(A)$. This is true for the unitary real form (defined below) and the split real form (see [78]). Moreover, in these cases $i(G^0) = G^0(A)$ is generated by the groups $G_\alpha^0(A) := G^0(A) \cap G_\alpha$. We do not know whether either of these statements remains true for arbitrary real forms.

In Section 3 we obtained a presentation of compact Lie groups, i.e. unitary real forms of complex semisimple Lie groups, as Phan amalgams. Recall that a unitary form of a complex semi-simple Lie group G is the fixed point set $K = G^\tau$ of the unique involution τ on G whose derivative restricts to negative conjugated transpose on each rank 1 subalgebra. Such an involution can also be defined for complex Kac-Moody groups, see [51]. We recall the construction here to fix our notation: Let ω be the Chevalley involution of $\mathfrak{g}(A)$ (see [49, p. 7]) and let $\bar{\omega}$ be the compact involution $\bar{\omega}(X) := \overline{\omega(X)}$. This involution restricts to

an involution $\bar{\omega}$ of $\mathfrak{g}'(A)$ and the subalgebra $\mathfrak{k}(A) := \{X \in \mathfrak{g}'(A) : \bar{\omega}(X) = X\} \subseteq \mathfrak{g}'(A)$ is a real form (with associated involution $\bar{\omega}$). As $\bar{\omega}$ restricts to negative conjugated transpose on each of the rank one subalgebras $\mathfrak{g}(\alpha)$, we may call $\mathfrak{k}(A)$ the *unitary form* of $\mathfrak{g}'(A)$ in accordance with the notion for complex semisimple Lie algebras.

The real form $K(A)$ of $G(A)$ corresponding to the unitary form $\mathfrak{k}(A)$ is also called the *unitary form* of $G(A)$. The involution $\bar{\omega} : G(A) \rightarrow G(A)$ obtained from integrating $\bar{\omega}$ fixes every rank one subgroup and satisfies $K_\alpha := G_\alpha^{\bar{\omega}} = G_\alpha \cap K(A)$. Moreover, $K(A)$ is generated by the groups K_α for real α . If we restrict the inclusion maps $\varphi_\alpha : \mathrm{SL}_2(\mathbb{C}) \rightarrow G_\alpha$ to $\mathrm{SU}_2(\mathbb{C})$, then we obtain isomorphisms $\psi_\alpha : \mathrm{SU}_2(\mathbb{C}) \rightarrow K_\alpha$. By Corollary 6.7, the Kac-Peterson topology on $G(A)$ induces a k_ω -topology on $K(A)$ which we also call the *Kac-Peterson topology* on $K(A)$ and denote by the same letter τ_{KP} . This topology can again be characterized by a universal property:

Proposition 6.9 *If $\Pi = \{\alpha_1, \dots, \alpha_N\}$ is any choice of simple roots for $\mathfrak{g}(A)$, then the Kac-Peterson topology τ_{KP} on $K(A)$ is the final group topology with respect to the inclusions $\psi_{\alpha_i} : \mathrm{SU}_2(\mathbb{C}) \rightarrow K(A)$, $i = 1, \dots, N$.*

Proof. Equip the groups K_{α_i} with their compact connected Lie group topology and let τ_0 be the final group topology with respect to the maps ψ_{α_i} or, equivalently, the inclusions $K_{\alpha_i} \hookrightarrow K(A)$. Note that the maps ψ_{α_i} are continuous with respect to τ_{KP} and thus τ_0 is finer than τ_{KP} ; in particular, τ_0 is Hausdorff. Similarly as in the proof of Theorem 6.3, we associate to each word $I = (i_1, \dots, i_k)$ over $\{1, \dots, N\}$ a map

$$\psi_I : K_{\alpha_{i_1}} \times \dots \times K_{\alpha_{i_k}} \rightarrow K(A), \quad (g_1, \dots, g_k) \mapsto g_1 \cdots g_k,$$

whose image is denoted K_I . The maps ψ_I are continuous both with respect to τ_0 and τ_{KP} . As these topologies are Hausdorff and $K_{\alpha_{i_1}} \times \dots \times K_{\alpha_{i_k}}$ is compact, ψ_I is a quotient map for both τ_0 and τ_{KP} . In particular, τ_0 and τ_{KP} coincide on each of the subsets K_I . As in the proof of Theorem 6.3, we see that $(K(A), \tau_0) = \varinjlim K_I$. On the other hand, $(K(A), \tau_{KP}) = \varinjlim (G_I \cap K(A))$. As $K_I \subseteq G_I \cap K(A)$ and the two topologies coincide on every K_I , it suffices to show that for each word $I \in \mathcal{W}_N$, there exists $J \in \mathcal{W}_N$ such that $G_I \cap K(A) \subseteq K_J$. This is essentially contained in the proof of Proposition 5.1 of [51] but not stated explicitly, whence we elaborate the argument here: Denote by s_j the root reflection at α_j . Then $S = \{s_1, \dots, s_N\}$ is a system of Coxeter generators for W . Let us call a word $I = (i_1, \dots, i_k)$ *special* if $(s_{i_1}, \dots, s_{i_k})$ is a reduced expression for the Weyl group element $w(I) := s_{i_1} \cdots s_{i_k} \in W$. Let us compute $G_I \cap K(A)$ in this case: By the Bruhat decomposition for $\mathrm{SL}_2(\mathbb{C})$, we have $G_{\alpha_i} \subseteq B^+ \cup B^+ s_i B^+$. It then follows from the general theory of Tits systems that

$$G_I \subseteq (B^+ \cup B^+ s_{i_1} B^+) \cdots (B^+ \cup B^+ s_{i_k} B^+) \subseteq \bigcup_{w \leq w(I)} B^+ w B^+,$$

where the inequality $w \leq w(I)$ is understood with respect to the Bruhat ordering of the Coxeter system (W, S) . Then

$$G_I \cap K(A) \subseteq \bigcup_{w \leq w(I)} K(A) \cap B^+ w B^+.$$

By Part (c) of Proposition 5.1 in [51], we have $K(A) \cap B^+ w B^+ \subseteq K_I \cdot T$ for each $w \leq w(I)$, where T is the compact torus defined by $T := H \cap K(A)$ and $H = \bigcap_{\alpha \in \Delta^{re}} N_{G(A)}(U_\alpha)$ as above. This shows that

$$G_I \cap K(A) \subseteq K_I \cdot T.$$

Now $T \subseteq K_1 \cdots K_N$ and thus if we define J as the concatenation of I with $(1, \dots, N)$, then $G_I \cap K(A) \subseteq K_J$. If $I \in \mathcal{W}_N$ is arbitrary, then this argument can be modified as follows: Let \mathcal{W}_I denote the set of special subwords of I . This is a finite subset of \mathcal{W}_N and thus there exists a special word $I' \in \mathcal{W}_N$ such that

$$(\forall J \in \mathcal{W}_I) \quad w(I') \geq w(J).$$

Then as before the theory of Tits systems implies

$$G_I \subseteq \bigcup_{w \leq w(I')} B^+ w B^+$$

and thus if we define J as the concatenation of I' with $(1, \dots, N)$, the same argument as before shows $G_I \cap K(A) \subseteq K_J$, finishing the proof. \square

Let us now see that the rank two subgroups are sufficient for a characterization of the unitary form $K(A)$. Adapting [37, Definition 4.3], we declare:

Definition 6.10 Let Δ be a two-spherical Dynkin diagram with set of labels Π , i.e., a Dynkin diagram whose induced subdiagrams on at most two nodes are spherical. Let G be a topological group. A *topological weak Phan system of type Δ over \mathbb{C} in G* is a family $(\overline{K}_j)_{j \in \Pi}$ of subgroups of G with the following properties:

twP1 G is generated by $\bigcup_{j \in \Pi} \overline{K}_j$;

twP2 \overline{K}_j is isomorphic as a topological group to a central quotient of a simply connected, compact, semisimple Lie group of rank one, for each $j \in \Pi$;

twP3 For all $i \neq j$ in Π , the subgroup $\overline{K}_{ij} := \langle \overline{K}_i \cup \overline{K}_j \rangle$ of G is isomorphic as a topological group to a central quotient of the simply connected, compact, semisimple Lie group of rank two belonging to the rank two subdiagram of Δ on the nodes i and j ;

twP4 $(\overline{K}_i, \overline{K}_j)$ or $(\overline{K}_j, \overline{K}_i)$ is a standard pair in \overline{K}_{ij} as defined in the paragraph before Theorem 3.1, for all $i \neq j$ in Π .

Definition 6.11 Let Δ be a two-spherical Dynkin diagram with set of types Π . An amalgam $\mathcal{A} = (\overline{K}_j)_{j \in (\Pi)_1 \cup (\Pi)_2}$ is called a *Phan amalgam of type Δ over \mathbb{C}* , if for all $\alpha, \beta \in \Pi$ the group $\overline{K}_{\alpha\beta}$ is a central quotient of a simply connected compact semisimple Lie group of type indicated by α, β , and if its morphisms are maps $\overline{K}_\alpha \hookrightarrow \overline{K}_{\alpha\beta}$ which embed $(\overline{K}_\alpha, \overline{K}_\beta)$ as a standard pair into $\overline{K}_{\alpha\beta}$ for $\alpha \neq \beta$.

A weak Phan system clearly gives rise to a Phan amalgam. The converse is in general not true. However, if there exists an enveloping group G of a Phan amalgam \mathcal{A} into which \mathcal{A} embeds, then the image of \mathcal{A} in G is a weak Phan system of G . Phan amalgams admitting such an enveloping group are called *strongly noncollapsing*.

We have encountered examples of weak Phan systems in Section 3. More generally, let A be a generalized Cartan matrix of two-spherical and irreducible type (W, S) and finite rank $|S|$ and consider the unitary form $K(A)$ of the complex Kac-Moody group $G(A)$, which we turn into a k_ω -group via the Kac-Peterson topology τ_{KP} . A topological weak Phan system for $(K(A), \tau_{KP})$ can be constructed as follows: The type of the Phan system is the Dynkin diagram Δ of type A . For any fundamental root $\alpha \in \Pi$ we put $\overline{K}_\alpha := K_\alpha$, which is identified as a topological group with $\mathrm{SU}_2(\mathbb{C})$ as above. By construction, $(\overline{K}_\alpha)_{\alpha \in \Pi}$ is a topological weak Phan system for G and $(K(A), \tau_{KP})$ is an enveloping group of the corresponding topological Phan amalgam. This topological Phan amalgam is called the *topological standard Phan amalgam defined by $(K(A), \tau)$* . We will show later that, if the Dynkin diagram Δ of $K(A)$ is a tree, then there essentially exists a unique Phan amalgam associated to Δ . On the other hand, if Δ admits cycles, this cannot be expected in view of [86, Section 6.5] and [68].

We want to prove that any topological group G with a topological weak Phan system of type Δ over \mathbb{C} , where Δ is a tree, carries a refined k_ω -topology τ' such that (G, τ') is a central quotient of some $(K(A), \tau_{KP})$. As a first step in this direction, we show that $(K(A), \tau_{KP})$ is a colimit of the amalgam defined by its topological Phan system.

Theorem 6.12 *Let $K(A)$ the unitary form of some complex Kac-Moody group $G(A)$ associated with a symmetrizable generalized Cartan matrix of finite size and two-spherical type (W, S) , let Δ^{re} be the set of real roots, and Π be a system of fundamental roots. Let \mathbb{I} be the small category with objects $(\Pi)_1 \cup (\Pi)_2$ and morphisms $\{\alpha\} \rightarrow \{\alpha, \beta\}$, for all $\alpha, \beta \in \Pi$, and let $\delta: \mathbb{I} \rightarrow \mathbb{KOG}$ be a diagram with $\delta(\{\alpha\}) = K_\alpha$, $\delta(\{\alpha, \beta\}) = K_{\alpha\beta}$ and $\delta(\{\alpha\} \rightarrow \{\alpha, \beta\}) = (K_\alpha \hookrightarrow K_{\alpha\beta})$.*

Then $((K(A), \tau_{KP}), (\iota_i)_{i \in (\Pi)_1 \cup (\Pi)_2})$, where ι_i is the natural inclusion map, is a colimit of δ

- (a) *in the category \mathbb{G} of abstract groups;*
- (b) *in the category \mathbb{HTG} of Hausdorff topological groups;*
- (c) *in the category \mathbb{KOG} of k_ω groups.*

Proof. (a) is proved exactly as Theorem 3.1, (b) follows from Proposition 6.9 and Part (c) follows from (b) and Corollary 5.10. \square

We now state the main result of this section. It is a general topological version of the main result of [37].

Theorem 6.13 (Topological Phan-type Theorem) *Let Δ be a two-spherical finite Dynkin tree-diagram with generalized Coxeter matrix A and let (G, \mathcal{O}) be a Hausdorff topological group admitting a topological weak Phan system of type Δ over \mathbb{C} . Then the topology on G can be refined to a group topology making G a k_ω -group isomorphic to a central quotient of the unitary form $(K(A), \tau_{KP})$ of a complex Kac-Moody group $G(A)$ equipped with its Kac-Peterson topology.*

If Δ is spherical, then the refined topology coincides with the original topology, and (G, \mathcal{O}) is a compact Lie group isomorphic to a central quotient of a simply connected compact semisimple Lie group whose universal complexification is a simply connected complex semisimple Lie group of type Δ .

Notice that we do *not* claim that the group G in Theorem 6.13 is a topological quotient of the model $(K(A), \tau_{KP})$, or a k_ω -group. Both statements are in general only true after refining the topology of G . To illustrate the necessity of this refinement we consider affine Kac-Moody groups:

Example 6.14 Let Δ_0 be a spherical Dynkin diagram and A_0 the associated Coxeter matrix. Then $G_0 := G(A_0)$ is a finite-dimensional complex Lie group. Let us assume that G is simple and not of type A_n . Then the affine extension Δ of Δ_0 is a tree and thus Theorem 6.13 applies. Denote by A the generalized Coxeter matrix of Δ and abbreviate $G := G(A)$, $K := K(A)$. We can consider G_0 as an algebraic group over \mathbb{C} and define the group $L_0G_0 := G_0(\mathbb{C}[[t]][t^{-1}])$ of $\mathbb{C}[[t]][t^{-1}]$ -rational points (cf. [62, Section 13.2]). The natural \mathbb{C}^\times -action on $\mathbb{C}[[t]][t^{-1}]$ by $(z.f)(t) := f(zt)$ induces a homomorphism $\mathbb{C}^\times \rightarrow \text{Aut}(L_0G_0)$ and we write \tilde{L}_0G_0 for the corresponding semidirect product. By [62, Definition 7.4.1 and Theorem 13.2.8], there is a group homomorphism $q_0: G \rightarrow \tilde{L}_0(G_0)/C_0$ with central kernel, where C_0 denotes the center of \tilde{L}_0G_0 . (Notice that our group G coincides with Kumar's \mathcal{G}_{\min}). Now L_0G_0 embeds naturally into the group $LG_0 := C^\infty(\mathbb{C}^\times, G_0)$ and similarly \tilde{L}_0G_0 embeds into

$$\tilde{L}G_0 := C^\infty(\mathbb{C}^\times, G_0) \ltimes \mathbb{C}^\times,$$

where the \mathbb{C}^\times -action is given by precomposition. If we equip $C^\infty(\mathbb{C}^\times, G_0)$ with the topology of uniform convergence of all derivatives and $\tilde{L}G_0$ with the corresponding product topology, then the latter becomes a metrizable topological group. The same is true of the quotient $\tilde{L}_0(G_0)/C$ by the center C . Denote by (\tilde{K}, τ) the image of K under the homomorphism

$$q: G \xrightarrow{q_0} \tilde{L}_0(G_0)/C_0 \hookrightarrow \tilde{L}G_0/C$$

equipped with the topology induced from the metrizable topology on $\tilde{L}G_0/C$ introduced above. Denote by $(K_j)_{j \in \Pi}$ the standard Phan system in K and write $\overline{K_j}$ for the image of K_j under q . We observe that $q|_{K_j}$ is continuous and that $\overline{K_j}$ is a central quotient of K_j (since q has central kernel). These facts combine to show that $(\overline{K_j})_{j \in \Pi}$ is a topological weak Phan system for (\widehat{K}, τ) . On the other hand, since \widehat{K} is metrizable and not locally compact, it cannot be k_ω by Proposition 4.8. However, the topology τ is easily refined to a k_ω -topology τ_0 (we can take the topology making $q|_K$ a quotient map).

Notice that one can produce a lot of non- k_ω topologies with weak topological Phan systems on the group \widehat{K} by using various topologies on $C^\infty(\mathbb{C}^\times, G_0)$. These topologies also lead to a multitude of topologies on the flag variety (or building) of $G(A)$ on which \widehat{K} acts transitively. We would like to advertise the topology induced from the k_ω -topology on \widehat{K} (or, equivalently, $G(A)$) as the most canonical topology on the flag variety. It has the important advantage that the Schubert decomposition of the flag variety becomes an honest CW decomposition. Moreover, with this topology the geometric realization of the associated topological building will be contractible. The latter property is a crucial ingredient in the Kramer-Mitchell-Quillen proof of Bott periodicity ([61], [65]). Finally, let us mention that the k_ω -topology that we call Kac-Petersen topology occurs under various names in the literature. In [62], it is called the *analytic topology* (more precisely, the analytic topology on Kumar's group \mathcal{G} restricts to the Kac-Petersen topology on our group $G(A) \subseteq \mathcal{G}$), to distinguish it from the Zariski topology.

Definition 6.15 Let $\mathcal{A} = (P_1 \xleftarrow{\iota_1} (P_1 \cap P_2) \xrightarrow{\iota_2} P_2)$ and $\mathcal{A}' = (P'_1 \xleftarrow{\iota'_1} (P'_1 \cap P'_2) \xrightarrow{\iota'_2} P'_2)$ be amalgams consisting of abstract groups. The amalgams \mathcal{A} and \mathcal{A}' are of the same *type* if there exist isomorphisms $\varphi_i: P_i \rightarrow P'_i$ such that $(\varphi_i \circ \iota_i)(P_1 \cap P_2) = \iota'_i(P'_1 \cap P'_2)$ for $i = 1, 2$. If, in addition, there exists an isomorphism $\varphi_{12}: P_1 \cap P_2 \rightarrow P'_1 \cap P'_2$ such that the diagram

$$\begin{array}{ccccc}
 & & P_1 & \xrightarrow{\varphi_1} & P'_1 \\
 & \nearrow \iota_1 & & & \nearrow \iota'_1 \\
 P_1 \cap P_2 & \xrightarrow{\varphi_{12}} & P'_1 \cap P'_2 & & \\
 & \searrow \iota_2 & & & \searrow \iota'_2 \\
 & & P_2 & \xrightarrow{\varphi_2} & P'_2
 \end{array}$$

commutes, then the amalgams \mathcal{A} and \mathcal{A}' are called *isomorphic*. For sake of brevity, we denote an isomorphism between the amalgams \mathcal{A} and \mathcal{A}' by the triple $(\varphi_1, \varphi_{12}, \varphi_2)$.

In case the amalgams consist of topological groups, the isomorphisms involved are assumed to be topological.

Lemma 6.16 (Goldschmidt's Lemma; cf. [31]) *Let $\mathcal{A} = (P_1 \leftarrow (P_1 \cap P_2) \rightarrow P_2)$ be an amalgam consisting of topological groups, let $A_i = \text{Stab}_{\text{Aut}(P_i)}(P_1 \cap P_2)$ for $i = 1, 2$, and let $\alpha_i: A_i \rightarrow \text{Aut}(P_1 \cap P_2)$ be homomorphisms mapping $a \in A_i$ onto its restriction*

to $P_1 \cap P_2$. Then there is a one-to-one correspondence between isomorphism classes of amalgams of the same type as \mathcal{A} and $\alpha_2(A_2)\text{-}\alpha_1(A_1)$ double cosets in $\text{Aut}(P_1 \cap P_2)$. In other words, there is a one-to-one correspondence between the different isomorphism types of amalgams $P_1 \hookleftarrow (P_1 \cap P_2) \hookrightarrow P_2$ and the double cosets $\alpha_2(A_2) \backslash \text{Aut}(P_1 \cap P_2) / \alpha_1(A_1)$.

Proof. By Definition 6.15, any amalgam of the same type as $P_1 \xleftarrow{\iota_1} (P_1 \cap P_2) \xrightarrow{\iota_2} P_2$ is isomorphic to an amalgam of the form $P_1 \xleftarrow{\iota_1 \circ \beta_1} (P_1 \cap P_2) \xrightarrow{\iota_2 \circ \beta_2} P_2$, where β_1 and β_2 are automorphisms of $P_1 \cap P_2$. Such an amalgam is isomorphic to $P_1 \xleftarrow{\iota_1} (P_1 \cap P_2) \xrightarrow{\iota_2 \circ \beta_2 \circ \beta_1^{-1}} P_2$ by the amalgam isomorphism $(\text{id}, \beta_1, \text{id})$. It remains to decide when two amalgams $P_1 \xleftarrow{\iota_1} (P_1 \cap P_2) \xrightarrow{\iota_2 \circ \gamma_1} P_2$ and $P_1 \xleftarrow{\iota_1} (P_1 \cap P_2) \xrightarrow{\iota_2 \circ \gamma_2} P_2$ are isomorphic. That is, we want to find an amalgam isomorphism $(\varphi_1, \varphi_{12}, \varphi_2)$ such that $\iota_1 \circ \varphi_{12} = \varphi_1 \circ \iota_1$ and $\iota_2 \circ \gamma_2 \circ \varphi_{12} = \varphi_2 \circ \iota_2 \circ \gamma_1$. The former equality says $\varphi_{12} \in \alpha_1(A_1)$ while the latter equality says $\gamma_2 \circ \varphi_{12} \circ \gamma_1^{-1} \in \alpha_2(A_2)$. Therefore an isomorphism between these two amalgams exists if and only if $\alpha_1(A_1) \cap \gamma_2^{-1} \alpha_2(A_2) \gamma_1 \neq \emptyset$. The latter is equivalent to $\alpha_2(A_2) \gamma_1 \alpha_1(A_1) = \alpha_2(A_2) \gamma_2 \alpha_1(A_1)$. \square

Definition 6.17 A Phan amalgam $(\overline{K}_{\alpha\beta})_{\alpha, \beta \in \Pi}$, and any subamalgam of a Phan amalgam, is called *unambiguous* if every $\overline{K}_{\alpha\beta}$ is isomorphic to the corresponding $K_{\alpha\beta}$.

Proposition 6.18 Every topological Phan amalgam $\overline{\mathcal{A}} = (\overline{K}_J)_{J \in \binom{\Pi}{1} \cup \binom{\Pi}{2}}$ has an unambiguous covering $\mathcal{A} = (G_J)_{J \in \binom{\Pi}{1} \cup \binom{\Pi}{2}}$ that is unique up to equivalence of coverings. Furthermore, every strongly noncollapsing Phan amalgam $\overline{\mathcal{A}}$ has a unique (up to equivalence of coverings) unambiguous strongly noncollapsing covering \mathcal{A} .

Proof. We proceed by induction on $|S|$, where S is a subset of $\binom{\Pi}{1} \cup \binom{\Pi}{2}$ with the property that for each $J \in S$, the inclusion $\emptyset \neq J' \subseteq J$ implies $J' \in S$. We order S by inclusion, i.e., the maximal elements of S are those sets which are not properly contained in another element of S . In particular, if J is a maximal element of S , then $S \setminus \{J\}$ is again a subset of $\binom{\Pi}{1} \cup \binom{\Pi}{2}$ with the property that for each $J' \in S \setminus \{J\}$, the inclusion $\emptyset \neq J'' \subseteq J'$ implies $J'' \in S \setminus \{J\}$. Let $\overline{\mathcal{A}}_S = (\overline{K}_J)_{J \in S}$ with morphisms $\overline{\iota}_{J', J} : \overline{K}_{J'} \rightarrow \overline{K}_J$ for $J' \subsetneq J$. The case $S = \emptyset$, which vacuously yields an unambiguous amalgam, serves as basis of the induction.

Suppose that S is non-empty and that for every subset $S' \subsetneq S$ the conclusion of the proposition holds. Let J be a maximal element of S , define $S' = S \setminus \{J\}$, and consider $\overline{\mathcal{A}}_{S'} = (\overline{K}_{J'})_{J' \in S'}$. By the inductive hypothesis, there is a unique unambiguous covering amalgam $(\mathcal{A}_{S'} = (K_{J'})_{J' \in S'}, \pi')$ of $\overline{\mathcal{A}}_{S'}$. By the definition of a Phan amalgam, there is a quotient map ψ from K_J onto \overline{K}_J . In case $|J| = 1$ define \mathcal{A}_S as the union of $\mathcal{A}_{S'}$ and $\{K_J\}$ without additional morphisms. Clearly, \mathcal{A}_S is an unambiguous covering of $\overline{\mathcal{A}}_S$. In case $|J| = 2$ assume $J = \{\alpha, \beta\}$. For $J' \subsetneq J$ let $\iota_{J', J} : K_{J'} \hookrightarrow K_J$ be the natural inclusion map. The function ψ maps $\mathcal{G} = (\iota_{J', J}(K_{J'}))_{J' \subsetneq J}$ onto $(\overline{\iota}_{J', J}(\overline{K}_{J'}))_{J' \subsetneq J}$. Indeed, the standard pair $(K_{\{\alpha\}}, K_{\{\beta\}})$ of $K_{\{\alpha, \beta\}}$ is mapped by ψ onto the standard pair $(\overline{K}_{\{\alpha\}}, \overline{K}_{\{\beta\}})$ of $\overline{K}_{\{\alpha, \beta\}}$, cf. 6.11 and the paragraph before 3.1. Furthermore, by the inductive hypothesis there is

an amalgam isomorphism $\varphi: (\mathcal{G}, \pi'|_{\mathcal{G}}) \rightarrow (\mathcal{G}, \psi|_{\mathcal{G}})$ such that $\psi \circ \varphi = \pi'|_{\mathcal{G}}$. Therefore we can define an unambiguous covering \mathcal{A}_S of $\overline{\mathcal{A}}_S$ as the union of $\mathcal{A}_{S'}$ and $\{K_J\}$ with the additional morphisms $\iota_{J',J} \circ \varphi$ for $J' \subsetneq J$. This completes the proof of the existence of an unambiguous covering \mathcal{A}_S .

It remains to prove the uniqueness of the unambiguous covering \mathcal{A}_S that we have just constructed. Let $\mathcal{B}_S = (B_J)_{J \in S}$ and $\mathcal{C}_S = (C_J)_{J \in S}$ be two such coverings with amalgam homomorphism π_1 , respectively π_2 onto \mathcal{A} . By the inductive hypothesis, there exists an isomorphism $\varphi: \mathcal{B}_{S'} \rightarrow \mathcal{C}_{S'}$ with $\pi_1|_{\mathcal{B}_{S'}} = \pi_2 \circ \varphi$. In order to extend φ to B_J , we have to deal with two cases: First, let $J = \{\alpha, \beta\}$ where α and β are orthogonal roots. In this case, $B_{\alpha\beta} \cong C_{\alpha\beta} \cong K_{\alpha\beta}$ are isomorphic to a direct product of $B_\alpha \cong C_\alpha \cong K_\alpha$ and $B_\beta \cong C_\beta \cong K_\beta$. Clearly, φ is already known on B_α and B_β , and so φ extends uniquely to $B_{\alpha\beta}$. This extension, also denoted φ , is a well-defined amalgam isomorphism from \mathcal{B}_S to \mathcal{C}_S , and furthermore $\pi_1 = \pi_2 \circ \varphi$ holds. In the second case, $B_J \cong C_J \cong K_J$ is isomorphic to a simply connected compact almost simple Lie group of rank one or two. By the universality of the covering $\pi_1: B_J \rightarrow \overline{K}_J$, as B_J is simply connected, there exists a unique homeomorphism $\psi: B_J \rightarrow C_J$ such that $\pi_1 = \pi_2 \circ \psi$. Consider the map $\alpha: \overline{K}_J \rightarrow \overline{K}_J$, $u \mapsto (\pi_2 \circ \psi \circ \pi_1^{-1})(u)$. This map α is a well-defined automorphism of \overline{K}_J , because the cosets of the kernel of π_1 are mapped by ψ to cosets of the kernel of π_2 . Every automorphism of \overline{K}_J , in particular α , is continuous by a corollary of Gotô's Commutator Theorem (see [47, Corollary 6.56]) and van der Waerden's Continuity Theorem (cf. [47, Theorem 5.64]). By [58], α lifts to a unique continuous automorphism of C_J (see also [46]). In other words, there is a unique (continuous) automorphism β of C_J such that $\pi_2 \circ \beta = \alpha \circ \pi_2$. Define $\theta: B_J \rightarrow C_J$, $\theta(b) = (\beta^{-1} \circ \psi)(b)$. First of all, by definition we have $\pi_1|_{B_J} = \pi_2 \circ \theta$, as

$$\begin{aligned} \pi_2 \circ \theta &= \pi_2 \circ \beta^{-1} \circ \psi \\ &= \alpha^{-1} \circ \pi_2 \circ \psi \\ &= \pi_1|_{B_J} \circ \psi^{-1} \circ \pi_2^{-1}|_{\overline{K}_J} \circ \pi_2 \circ \psi \\ &= \pi_1|_{B_J}. \end{aligned}$$

Second, for every $J' \subseteq J$, we have that $\theta^{-1} \circ \varphi|_{B_{J'}}$ is a lifting to $B_{J'}$ of the identity automorphism of $\overline{K}_{J'}$ and, by the above, it is the identity. This holds due to $\theta^{-1} \circ \varphi|_{B_{J'}} = \psi^{-1} \circ \beta \circ \varphi|_{B_{J'}}$ and the following, considered on $B_{J'}/\ker(\pi_1|_{B_{J'}})$:

$$\begin{aligned} \psi^{-1} \circ \pi_2|_{C_{J'}}^{-1} \circ \alpha \circ \pi_2 \circ \varphi|_{B_{J'}} &= \psi^{-1} \circ \pi_2|_{C_{J'}}^{-1} \circ \pi_2 \circ \psi \circ \pi_1|_{B_{J'}}^{-1} \circ \pi_2 \circ \varphi|_{B_{J'}} \\ &= \pi_1|_{B_{J'}}^{-1} \circ \pi_2 \circ \varphi|_{B_{J'}} \\ &= \text{id}. \end{aligned}$$

Thus φ and θ agree on every subgroup $B_{J'}$, which allows us to extend φ to all of \mathcal{B}_S by defining it on B_J as θ . Finally, if $\overline{\mathcal{A}}$ is strongly noncollapsing, so is its unique unambiguous covering \mathcal{A} , finishing the proof. \square

Proposition 6.19 *Let $n \geq 2$, and let \mathcal{A} be a tree-like strongly noncollapsing unambiguous irreducible Phan amalgam of rank n . Then \mathcal{A} is unique up to isomorphism, i.e., \mathcal{A} is isomorphic to a standard Phan amalgam.*

Proof. Let $\mathcal{A} = (K_J)_{J \in \binom{\Pi}{1} \cup \binom{\Pi}{2}}$ be a tree-like unambiguous strongly noncollapsing irreducible Phan amalgam of rank at least two. We proceed by induction on n . The amalgams of rank two are unique by definition.

Rank three

Let $\Pi = \{\alpha, \beta, \gamma\}$ and let \mathcal{A} and \mathcal{A}' be the amalgams

$$\begin{array}{ccc} K_\alpha & \longrightarrow & K_{\alpha\beta} \\ & \searrow \nearrow & \\ K_\beta & & K_{\alpha\gamma} \\ & \searrow \nearrow & \\ K_\gamma & \longrightarrow & K_{\beta\gamma} \end{array} \quad \text{respectively} \quad \begin{array}{ccc} K'_\alpha & \longrightarrow & K'_{\alpha\beta} \\ & \searrow \nearrow & \\ K'_\beta & & K'_{\alpha\gamma} \\ & \searrow \nearrow & \\ K'_\gamma & \longrightarrow & K'_{\beta\gamma} \end{array}$$

According to Goldschmidt's Lemma (Lemma 6.16), the amalgams $\mathcal{B} = (K_{\alpha\beta} \leftarrow K_\beta \hookrightarrow K_{\beta\gamma})$ and $\mathcal{B}' = (K'_{\alpha\beta} \leftarrow K'_\beta \hookrightarrow K'_{\beta\gamma})$ are isomorphic via some amalgam isomorphism ψ , because every automorphism of the group K_β is induced by some automorphism of the group $K_{\alpha\beta}$ (cf. [47, Theorem 6.73]). So clearly $\psi(K_\beta) = K'_\beta$.

The groups K_α and K_β form a standard pair in $K_{\alpha\beta}$, and hence $\psi(K_\alpha)$ and $K'_\beta = \psi(K_\beta)$ form a standard pair in $K'_{\alpha\beta} = \psi(K_{\alpha\beta})$. Therefore, since standard pairs are conjugate, there exists an automorphism of $K'_{\alpha\beta}$ that maps $\psi(K_\alpha)$ onto K'_α and that leaves $K'_\beta = \psi(K_\beta)$ invariant. Thus, we can assume $\psi(K_\alpha) = K'_\alpha$.

Define $D_\alpha = N_{K_\alpha}(K_\beta)$ and $D_\gamma = N_{K_\gamma}(K_\beta)$, where the groups K_β, K_α are considered as subgroups of $K_{\alpha\beta}$ and the groups K_β, K_γ are considered as subgroups of $K_{\beta\gamma}$. Since K_β and K_α form a standard pair in $K_{\alpha\beta}$, the group D_α is a maximal torus in K_α . For the same reason, D_γ is a maximal torus in K_γ . Moreover, let $D_\beta^\alpha = N_{K_\beta}(K_\alpha)$ and $D_\beta^\gamma = N_{K_\beta}(K_\gamma)$. Again, these are maximal tori in K_β , satisfying $D_\beta^i = C_{K_\beta}(D_i)$ for $i = \alpha, \gamma$. Let $\pi: \mathcal{A} \rightarrow G$ be an enveloping group of \mathcal{A} such that π is injective on every $\overline{K}_i, i \in \Pi$, which is possible since \mathcal{A} is strongly noncollapsing. Then $\pi(D_\alpha^i) = C_{\pi(K_\beta)}(\pi(D_i))$. Moreover, since $\pi(K_\beta)$ and $\pi(D_\gamma)$ are invariant under $\pi(D_\alpha) = N_{\pi(K_\alpha)}(\pi(K_\beta))$, the group $\pi(D_\beta^\gamma) = C_{\pi(K_\beta)}(\pi(D_\gamma))$ is invariant under $\pi(D_\alpha)$. So the maximal torus $\pi(D_\alpha)$ of $\pi(K_\alpha)$ leaves invariant the maximal tori $\pi(D_\beta^\alpha)$ and $\pi(D_\beta^\gamma)$ of $\pi(K_\beta)$. Analysis of the rank two group $\pi(K_{\alpha\beta})$ shows that $D_\beta^\alpha = D_\beta^\gamma$. Hence we can use the notation $D_\beta = D_\beta^\alpha = D_\beta^\gamma$. From $N_{K_\beta}(K_\alpha) = D_\beta$ it follows that $\psi(D_\beta) = D'_\beta := N_{K'_\beta}(K'_\alpha) = N_{K'_\beta}(K'_\gamma)$, because $\psi(K_\alpha) = K'_\alpha$ and $\psi(K_\beta) = K'_\beta$. Therefore we have identified a root system of type A_2, B_2 , or G_2 with respect to which the groups $K'_\beta = \psi(K_\beta), K'_\gamma, \psi(K_\gamma)$ occur as fundamental groups. By inspection (cf. [16,

Plate X]), there exist automorphisms of $K'_{\beta\gamma}$ mapping $\psi(K_\gamma)$ onto K'_γ and centralizing K'_β . Hence we can assume $\psi(K_\gamma) = K'_\gamma$.

Finally, we have $\psi(K_{\alpha\gamma}) = K'_{\alpha\gamma}$, because $K_{\alpha\gamma} \cong K_\alpha \times K_\gamma$ and $K'_{\alpha\gamma} \cong K'_\alpha \times K'_\gamma$.

Rank at least four

Let $|\Pi| \geq 4$ and let $\mathcal{A} = (K_J)_{J \in \binom{\Pi}{1} \cup \binom{\Pi}{2}}$ be a tree-like unambiguous strongly noncollapsing irreducible Phan amalgam. Then there exists a unique amalgam $\mathcal{B}_\mathcal{A}$ obtained by adding a colimit H_α of $\mathcal{A}_\alpha := (K_J)_{J \in \binom{\Pi \setminus \{\alpha\}}{1} \cup \binom{\Pi \setminus \{\alpha\}}{2}}$ and a colimit H_β of $\mathcal{A}_\beta := (K_J)_{J \in \binom{\Pi \setminus \{\beta\}}{1} \cup \binom{\Pi \setminus \{\beta\}}{2}}$ to the amalgam \mathcal{A} where α and β are perpendicular in Π . By the inductive hypothesis, both \mathcal{A}_α and \mathcal{A}_β are isomorphic to some standard Phan amalgam, so the colimits H_α and H_β are determined in Theorem 6.12.

In particular, if \mathcal{A} and \mathcal{A}' are tree-like unambiguous strongly noncollapsing irreducible Phan amalgams, then $H_\alpha \cong H'_\alpha$ and $H_\beta \cong H'_\beta$. For $\mathcal{B} := (K_J)_{J \in \binom{\Pi \setminus \{\alpha, \beta\}}{1} \cup \binom{\Pi \setminus \{\alpha, \beta\}}{2}}$ and $\mathcal{B}' := (K'_J)_{J \in \binom{\Pi \setminus \{\alpha, \beta\}}{1} \cup \binom{\Pi \setminus \{\alpha, \beta\}}{2}}$, again by induction $\mathcal{B} \cong \mathcal{B}'$, so — for a colimit H_0 of \mathcal{B} and a colimit H'_0 of \mathcal{B}' — the amalgams $H_\alpha \hookleftarrow H_0 \hookrightarrow H_\beta$ and $H'_\alpha \hookleftarrow H'_0 \hookrightarrow H'_\beta$ are of the same type; the isomorphism type of $H_0 \cong H'_0$ again is determined in Theorem 6.12. By [18, Theorem 8.2] and Goldschmidt's Lemma (Lemma 6.16), the amalgams $H_\alpha \hookleftarrow H_0 \hookrightarrow H_\beta$ and $H'_\alpha \hookleftarrow H'_0 \hookrightarrow H'_\beta$ are isomorphic under some map φ . The standard Phan amalgams $\varphi(\mathcal{B})$ and \mathcal{B}' in H'_0 correspond to two choices of a maximal torus of H'_0 , which are conjugate by the Iwasawa decomposition. So, correcting φ by an inner automorphism of H'_0 , we may assume that $\varphi(K_i) = K'_i$ for all $i \in \binom{\Pi \setminus \{\alpha, \beta\}}{1} \cup \binom{\Pi \setminus \{\alpha, \beta\}}{2}$. Also, by studying H'_α and H'_β , we have $\varphi(K_\alpha) = K'_\alpha$, $\varphi(K_\beta) = K'_\beta$ and $\varphi(K_{\alpha i}) = K'_{\alpha i}$, $\varphi(K_{\beta i}) = K'_{\beta i}$ for all $i \in \Pi \setminus \{\alpha, \beta\}$. It remains to realize that $K_{\alpha\beta}$ is the direct product of K_α and K_β , so that φ induces an isomorphism between \mathcal{A} and \mathcal{A}' . \square

Proof of Theorem 6.13. The weak Phan system of G gives rise to a strongly noncollapsing Phan amalgam \mathcal{A} , which by Proposition 6.18 is covered by a unique strongly noncollapsing unambiguous Phan amalgam $\hat{\mathcal{A}}$. This strongly noncollapsing unambiguous Phan amalgam $\hat{\mathcal{A}}$ is isomorphic to a standard Phan amalgam by Proposition 6.19. The claim then follows from Theorem 6.12 (and Theorem 3.1). \square

Let us now return to the Curtis-Tits Theorem which motivated our research on topological presentations resulting in this section of the current article. As we mentioned in the introduction, the Curtis-Tits Theorem was originally proved in [21], [83]. By a result of Abramenko and Mühlherr [1] (or, alternatively, Mühlherr [69]; see also [12], [38]), the Curtis-Tits Theorem follows from the simple connectedness of the opposites geometry that is associated to a twin building as follows. Given a twin building $\mathcal{T} = (\mathcal{B}_+, \mathcal{B}_-, \delta_*)$ consisting of the buildings $\mathcal{B}_+ = (\mathcal{C}_+, \delta^+)$ and $\mathcal{B}_- = (\mathcal{C}_-, \delta^-)$ and the codistance δ_* (cf. [88]), define the opposites chamber system as $\text{Opp}(\mathcal{T}) = \{(c_+, c_-) \in \mathcal{C}_+ \times \mathcal{C}_- \mid \delta_*(c_+, c_-) = 1\}$. (Chambers $x \in \mathcal{C}_+$ and $y \in \mathcal{C}_-$ with $\delta_*(x, y) = 1$ are called *opposite*, hence the name and notation; the concept of the opposites chamber system has been introduced in [87]). The

articles [1], [69] and Lemma 6.2 of the present paper imply the following theorem, which is a split version of Theorem 6.12 over the complex numbers.

Theorem 6.20 (Topological Curtis-Tits Theorem) *Consider a complex Kac-Moody group $G(A)$ associated with a symmetrizable generalized Cartan matrix of finite size and two-spherical type (W, S) . Let Δ^{re} be the set of real roots, and Π be a system of fundamental roots. Let \mathbb{I} be the small category with objects $\binom{\Pi}{1} \cup \binom{\Pi}{2}$ and morphisms $\{\alpha\} \rightarrow \{\alpha, \beta\}$, for all $\alpha, \beta \in \Pi$, and let $\delta: \mathbb{I} \rightarrow \mathbb{KOG}$ be a diagram with $\delta(\{\alpha\}) = G_\alpha$, $\delta(\{\alpha, \beta\}) = G_{\alpha\beta}$ and $\delta(\{\alpha\} \rightarrow \{\alpha, \beta\}) = (G_\alpha \hookrightarrow G_{\alpha\beta})$.*

Then $((G(A), \tau_{KP}), (\iota_i)_{i \in \binom{\Pi}{1} \cup \binom{\Pi}{2}})$, where ι_i is the natural inclusion map, is a colimit of δ

- (a) *in the category \mathbb{G} of abstract groups;*
- (b) *in the category \mathbb{HTG} of Hausdorff topological groups;*
- (c) *in the category \mathbb{KOG} of k_ω groups.*

Proof. (a) follows from [1] or [69], (b) follows from Lemma 6.2 and Part (c) follows from (b) and Corollary 5.10. \square

A classification of topological amalgams as for the unitary form leading to a split analogue of Theorem 6.13 over the complex numbers is not immediately possible by the methods presented in this paper, since van der Waerden's continuity theorem does not hold for complex Lie groups (as there exist discontinuous field automorphisms). In view of Shtern's generalization of this continuity theorem to arbitrary semisimple real Lie groups [80], a classification of amalgams over the field of real numbers is possible once the analogue of Lemma 6.2 and Proposition 6.9 has been proved for the split real form of $G(A)$. This leads to split analogues of Theorem 6.12 and Theorem 6.13 over the real numbers.

7 Duality of locally k_ω and almost metrizable groups

In this section, we show that the categories of locally k_ω abelian groups and almost metrizable abelian groups are dual to each other.

Recall that a Hausdorff space X is called *almost metrizable* if each $x \in X$ is contained in a compact set K which has a countable basis $(U_n)_{n \in \mathbb{N}}$ of neighbourhoods in X (see [6, Definition 1.22] or [75]). Thus each U_n is a subset of X having K in its interior, and for each subset $U \subseteq X$ such that $K \subseteq U^0$, there exists $n \in \mathbb{N}$ such that $U_n \subseteq U$. It is known that a Hausdorff topological abelian group G is almost metrizable if and only if G/K is metrizable for a compact subgroup $K \leq G$ (see [6, Proposition 2.20] or [75]).

We shall also need a related concept: A topological space is called *Čech complete* if it is the intersection of a sequence of open subsets of a compact space. It is known that a Hausdorff topological abelian group G is Čech complete if and only if G/K is complete

and metrizable for some compact subgroup $K \leq G$ (see [6, Corollary 2.21]). Hence every Čech complete abelian group is almost metrizable and complete.

The following notation will be used in our discussions of topological abelian groups. Given a Hausdorff topological abelian group G , we let $\eta_G: G \rightarrow G^{**}$, $\eta_G(x)(\xi) := \xi(x)$ be the evaluation homomorphism and say that G is *reflexive* if η_G is an isomorphism of topological groups. Given a continuous homomorphism $f: G \rightarrow H$, we let $f^*: H^* \rightarrow G^*$, $f^*(\xi) := \xi \circ f$ be the dual morphism. Then $f^{**} \circ \eta_G = \eta_H \circ f$. We write $\mathbb{T}_+ := \{z \in \mathbb{T}: \operatorname{Re}(z) \geq 0\}$. Given a subset $A \subseteq G$, we let $A^\circ := \{\xi \in G^*: \xi(A) \subseteq \mathbb{T}_+\}$ be its *polar* in G^* . If $B \subseteq G^*$, we write ${}^\circ B := \eta_G^{-1}(B^\circ)$ for its polar in G . Occasionally, we shall write $\langle \xi, x \rangle := \xi(x)$ for $x \in G$, $\xi \in G^*$.

Proposition 7.1 *If G is an abelian locally k_ω group, then G^* is Čech complete and hence almost metrizable and complete. Conversely, G^* is locally k_ω and complete, for each almost metrizable abelian group G .*

Proof. If G is an abelian locally k_ω group, then G has an open subgroup H which is a k_ω -group, by Proposition 5.3. Then $D := G/H$ is discrete. Let $i: H \rightarrow G$ be the inclusion map and $q: G \rightarrow D$ be the canonical quotient morphism. Then D^* is compact [47, Proposition 7.5 (i)] and H^* is complete and metrizable [6, Propositions 2.8 and 4.11]. Since i is an open embedding, i^* is a quotient morphism [10, Lemma 2.2 (d)]. Furthermore, q being surjective, q^* is injective and actually an embedding since D^* is compact. By [10, Lemma 1.4], we have a short exact sequence

$$\{1\} \rightarrow D^* \xrightarrow{q^*} G^* \xrightarrow{i^*} H^* \rightarrow \{1\}.$$

Then $K := q^*(D^*)$ is a compact subgroup of G^* and $G^*/K \cong H^*$ is complete and metrizable. Therefore G^* is Čech complete, using [6, Corollary 2.21].

Conversely, assume that G is almost metrizable. Then G/K is metrizable for a compact subgroup $K \subseteq G$. Let $i: K \rightarrow G$ be the inclusion map and $q: G \rightarrow G/K =: Q$ be the canonical quotient morphism. Since Q is metrizable, Q^* is a k_ω -group [6, Corollary 4.7] and complete (see [6, Proposition 4.11]). By [10, Lemma 2.5], $q^*: Q^* \rightarrow G^*$ is a continuous open homomorphism with compact kernel. Since q is surjective, q^* is injective and hence an open embedding. Thus G^* has the complete k_ω -group $q^*(Q^*)$ as an open subgroup. Hence G^* is complete, and Proposition 5.3 shows that G^* is locally k_ω . \square

Cf. [6, Proposition 5.20] for a related result.

Corollary 7.2 *Let G be a reflexive topological abelian group. Then G is almost metrizable if and only if G^* is locally k_ω (in which case G actually is Čech complete, and G^* is complete). Also, G is locally k_ω if and only if G^* is almost metrizable (in which case G actually is complete, and G^* is Čech complete).* \square

Remark 7.3 It is known that each abelian k_ω -group is complete [77]. Hence also each abelian locally k_ω group is complete (exploiting Proposition 5.3 (b)).

8 Dual groups of projective limits and direct limits

The dual groups of countable direct limits of abelian k_ω -groups and countable projective limits of metrizable abelian groups have been studied in [4]. In this section, we describe various generalizations of the results from [4]. In particular, we study the dual groups of countable direct limits of abelian, locally k_ω groups and countable projective limits of almost metrizable abelian groups.

Our first proposition generalizes [4, Proposition 3.1], the proof of which given in [4] requires that the limit maps q_i are surjective.⁴ Recall that a subgroup H of a topological abelian group G is called *dually embedded* if each character $\xi \in H^*$ extends to a character of G . It is *dually closed* if, for each $x \in G \setminus H$, there exists $\xi \in G^*$ such that $\xi|_H = 1$ and $\xi(x) \neq 1$.

Lemma 8.1 *Let $((G_i)_{i \in I}, (q_{ij})_{i \leq j})$ be a projective system of Hausdorff topological abelian groups and continuous homomorphisms $q_{ij}: G_j \rightarrow G_i$, with projective limit $(G, (q_i)_{i \in I})$. Then the following holds:*

- (a) *If $q_i(G)$ is dually embedded in G_i for each $i \in I$ (e.g., if each $q_i: G \rightarrow G_i$ has dense image), then each $\xi \in G^*$ is of the form $\xi = \theta \circ q_i$ for some $i \in I$ and $\theta \in G_i^*$. Hence G is dually embedded in $P := \prod_{i \in I} G_i$. If each q_i has dense image, then furthermore $(G^*, (q_i^*)_{i \in I}) = \varinjlim ((G_i^*)_{i \in I}, (q_{ij}^*)_{i \leq j})$ as an abstract group.*
- (b) *If each η_{G_i} is injective, then G is dually closed in $\prod_{i \in I} G_i$.*

Proof. (a) Since $\xi^{-1}(\mathbb{T}_+)$ is a 0-neighbourhood, there exists $i \in I$ and a 0-neighbourhood $U \subseteq G_i$ such that $q_i^{-1}(U) \subseteq \xi^{-1}(\mathbb{T}_+)$ (cf. [89, p. 23]). Set $D := q_i(G)$. Since \mathbb{T}^+ does not contain any non-trivial subgroups and $\xi(\ker(q_i)) \subseteq \mathbb{T}_+$, we deduce that $\ker(q_i) \subseteq \ker(\xi)$. Hence, there exists a homomorphism $\zeta: D \rightarrow \mathbb{T}$ such that $\zeta \circ q_i = \xi$. Since $\zeta(U \cap D) = \zeta(q_i(q_i^{-1}(U))) = \xi(q_i^{-1}(U)) \subseteq \mathbb{T}_+$, [53, Lemma 2.1] shows that ζ is continuous. Now D being dually embedded in G_i , the character ζ extends to a character $\theta: G_i \rightarrow \mathbb{T}$. Then $\theta \circ q_i = \xi$. Furthermore, ξ extends to the continuous homomorphism $\theta \circ \text{pr}_i: P \rightarrow \mathbb{T}$, where $\text{pr}_i: P \rightarrow G_i$ is the canonical projection. Hence G is dually embedded in P . By the preceding, $G^* = \bigcup_{i \in I} q_i^*(G_i^*)$. If we assume that each q_i has dense image, then each q_i^* is injective, entailing that $(G^*, (q_i^*)_{i \in I})$ is the asserted direct limit group.

(b) See [6, Lemma 5.28]. □

Note that if I is countable in the situation of Lemma 8.1 and each q_{ij} is surjective, then each q_i is surjective (as is well known).

Proposition 8.2 *Let $((G_i)_{i \in I}, (q_{ij})_{i \leq j})$ be a projective system of reflexive topological abelian groups, with projective limit $(G, (q_i)_{i \in I})$. If $q_i(G)$ is dually embedded in G_i for each $i \in I$ (e.g., if each q_i has dense image), then η_G is open and an isomorphism of groups.*

⁴The description of $\pi_1(\varinjlim G_\alpha)$ given in the displayed formula in [4, proof of Proposition 3.1] requires that the limit map to G_{α_k} is surjective.

Proof. By Lemma 8.1, G is dually closed and dually embedded in $\prod_{i \in I} G_i$. Each G_i being reflexive, also $P := \prod_{i \in I} G_i$ is reflexive (see [9, Proposition (14.11)] or [52]). Now η_P being an isomorphism of abstract groups and open, also η_G is an open isomorphism of groups by [6, Corollary 5.25] (a result due to Noble). \square

Remark 8.3 If η_G happens to be continuous in Proposition 8.2 (e.g., if G is a k -space), then G is reflexive. Recall that subgroups of nuclear abelian groups (as in [9, Definition 7.1]) are dually embedded by [9, Corollary 8.3]. Hence, if a projective limit G of reflexive nuclear groups G_i has a continuous evaluation homomorphism η_G , then G is reflexive. See [6], [9], and [25] for further information on nuclear groups.

As a corollary to Proposition 8.2, we obtain a generalization of [4, Theorem 3.2], where each G_n was assumed a metrizable, reflexive abelian topological group and where each $q_{n,m}$ (and hence q_n) was assumed surjective.

Corollary 8.4 *Let $((G_n)_{n \in \mathbb{N}}, (q_{n,m})_{n \leq m})$ be a projective sequence of reflexive topological abelian groups, with projective limit $(G, (q_n)_{n \in \mathbb{N}})$. If $q_n(G)$ is dually embedded in G_n (e.g., if q_n has dense image) and G_n is almost metrizable for each $n \in \mathbb{N}$, then G is reflexive.*

Proof. By Proposition 8.2, η_G is an open isomorphism of groups. As we assume that G_n is almost metrizable and reflexive, G_n is Čech complete (by Corollary 7.2). Thus also G is Čech complete (see [23, Corollary 3.9.9]) and hence a k -space, by [23, Theorem 3.9.5]. Therefore η_G is continuous by [6, Corollary 5.12] and hence an isomorphism of topological groups (cf. also [20]). \square

Recall that a topological abelian group G is called *locally quasi-convex* if it has a basis of 0-neighbourhoods U such that $U = {}^\circ(U^\circ)$. Given a topological abelian group G , there is a finest locally quasi-convex group topology \mathcal{O}_{lqc} on G which is coarser than the given topology; a basis of 0-neighbourhoods for \mathcal{O}_{lqc} is given by the bipolars ${}^\circ(U^\circ)$ of 0-neighbourhoods in G (see [6, Proposition 6.18]). Then $G_{\text{lqc}} := (G, \mathcal{O}_{\text{lqc}}) / \overline{\{0\}}$ is a locally quasi-convex group such that each continuous homomorphism from G to a Hausdorff locally quasi-convex group factors over G_{lqc} . In the following, $\varinjlim_{\text{lqc}} G_i$ denotes the direct limit of a direct system $\mathcal{S} = ((G_i)_{i \in I}, (\lambda_{ij})_{i \geq j})$ of locally quasi-convex Hausdorff abelian groups in the category of such groups. It can be obtained as

$$\varinjlim_{\text{lqc}} G_i = (\varinjlim G_i)_{\text{lqc}},$$

where $\varinjlim G_i = \varinjlim \mathcal{S}$ in the category of Hausdorff abelian groups (cf. [4, Proposition 4.2]).

Our next result generalizes [4, Theorem 4.3], which only applied to projective sequences $((G_n)_{n \in \mathbb{N}}, (q_{n,m})_{n \leq m})$ of metrizable, reflexive abelian topological groups such that each $q_{n,m}$ (and hence q_n) is surjective.

Proposition 8.5 *Let $\mathcal{S} := ((G_i)_{i \in I}, (q_{ij})_{i \leq j})$ be a projective system of reflexive topological abelian groups, with projective limit $(G, (q_i)_{i \in I})$. If each q_i has dense image, then $(G^*, (q_i^*)_{i \in I}) = \varinjlim ((G_i^*)_{i \in I}, (q_{ij}^*)_{i \leq j})$ holds in the category of locally quasi-convex Hausdorff topological abelian groups.*

Proof. Let \mathcal{T} be the compact-open topology on G^* . By Lemma 8.1, $G^* = \varinjlim G_i^*$ as an abstract group. Since (G^*, \mathcal{T}) is locally quasi-convex and the homomorphisms $q_i^*: G_i^* \rightarrow G^*$ are continuous, the topology \mathcal{O} making G^* the locally quasi-convex direct limit is finer than \mathcal{T} and thus $\lambda: (G^*, \mathcal{O}) \rightarrow (G^*, \mathcal{T})$, $\lambda(\xi) := \xi$ is continuous. It remains to show that every identity neighbourhood U in $H := (G^*, \mathcal{O})$ is also an identity neighbourhood in (G^*, \mathcal{T}) . We may assume that $U = {}^\circ(U^\circ)$. We write h_i for q_i^* , considered as a map $G_i^* \rightarrow H$; then h_i is continuous, $\lambda \circ h_i = q_i^*$, and $h_j \circ q_{ij}^* = h_i$ for $i \leq j$. The continuous homomorphisms $\eta_{G_i}^{-1} \circ h_i^*$ form a cone over the projective system \mathcal{S} since $q_{ij} \circ \eta_{G_j}^{-1} \circ h_j^* = \eta_{G_i}^{-1} \circ q_{ij}^{**} \circ h_j^* = \eta_{G_i}^{-1} \circ h_i^*$. Hence, there is a continuous homomorphism $\beta: H^* \rightarrow G$ such that $q_i \circ \beta = \eta_{G_i}^{-1} \circ h_i^*$ for all $i \in I$. Then $U^\circ \subseteq H^*$ is compact (cf. [9, Proposition 1.5]) and hence also $K := \beta(U^\circ) \subseteq G$ is compact. We claim that $U = K^\circ$; if we can show this, then U is also a 0-neighbourhood in (G^*, \mathcal{T}) and the proof is complete. To prove the claim, we note that $\langle \lambda(\xi), \beta(\theta) \rangle = \langle \theta, \xi \rangle$ for each $\theta \in H^*$ and $\xi \in H$. In fact, there is $i \in I$ such that $\xi = h_i(\zeta)$ for some $\zeta \in G_i^*$. Then

$$\begin{aligned} \langle \lambda(\xi), \beta(\theta) \rangle &= \langle \lambda(h_i(\zeta)), \beta(\theta) \rangle = \langle q_i^*(\zeta), \beta(\theta) \rangle = \langle \zeta, q_i(\beta(\theta)) \rangle \\ &= \langle \zeta, \eta_{G_i}^{-1}(h_i^*(\theta)) \rangle = \langle h_i^*(\theta), \zeta \rangle = \langle \theta, h_i(\zeta) \rangle = \langle \theta, \xi \rangle. \end{aligned}$$

Hence $\langle \theta, \xi \rangle = \langle \xi, \beta(\theta) \rangle$ for each $\theta \in U^\circ$ and $\xi \in G^*$ in particular, entailing that $U = {}^\circ(U^\circ) = K^\circ$. \square

As a corollary, we obtain a generalization of [4, Theorem 4.4].

Corollary 8.6 *Let $((G_n)_{n \in \mathbb{N}}, (q_{n,m})_{n \leq m})$ be a projective sequence of reflexive topological abelian groups, with projective limit $(G, (q_n)_{n \in \mathbb{N}})$. If q_n has dense image and G_n is almost metrizable for each $n \in \mathbb{N}$, then $\varinjlim_{\text{lqc}} G_n^*$ is reflexive.*

Proof. By Corollary 8.4, G is reflexive, whence also G^* is reflexive by [6, Proposition 5.9]. But $G^* = \varinjlim_{\text{lqc}} G_n^*$ by Proposition 8.5. \square

Part (a) of the next proposition generalizes [4, Corollary 2.2] (the corresponding statement for k_ω -groups), while part (b) generalizes [4, Theorem 4.5], which applied to strict direct sequences of abelian k_ω -groups G_n with $i_{n,m}(G_m)$ dually embedded in G_n (cf. also [4, Remark on p. 18]).

Proposition 8.7 *Let $\mathcal{S} := ((G_n)_{n \in \mathbb{N}}, (i_{n,m})_{n \geq m})$ be a direct sequence of abelian, locally k_ω groups, and $(G, (i_n)_{n \in \mathbb{N}})$ be its direct limit in the category of Hausdorff abelian groups.*

(a) *Then $(G^*, (i_n^*)_{n \in \mathbb{N}}) = \varprojlim ((G_n^*)_{n \in \mathbb{N}}, (i_{n,m}^*)_{n \geq m})$ as a topological group.*

- (b) If each G_n is reflexive and each $i_n^*: G^* \rightarrow G_n^*$ has dense image (which holds, for example, if $i_{n,m}$ is a topological embedding and $i_{n,m}(G_m)$ is dually embedded in G_n whenever $m \leq n$), then G_{lqc} is reflexive. Furthermore, $\gamma^*: (G_{\text{lqc}})^* \rightarrow G^*$ is a topological isomorphism, where $\gamma: G \rightarrow G_{\text{lqc}}$ is the canonical homomorphism.

Proof. (a) In the situation of (a), we have

$$(G^*, (i_n^*)_{n \in \mathbb{N}}) = \varprojlim ((G_n^*)_{n \in \mathbb{N}}, (i_{n,m}^*)_{n \geq m}) \quad (3)$$

in the category of sets, by the universal property of $G = \varinjlim G_n$ in the category of Hausdorff groups. Then clearly (3) also holds in the category of groups. Each map i_n^* being continuous, the compact-open topology \mathcal{O} on G^* is finer than the projective limit topology \mathcal{T} . If $K \subseteq G$ is compact, then $K \subseteq i_n(L)$ for some $n \in \mathbb{N}$ and some compact subset $L \subseteq G_n$ (Lemma 5.12) and thus $K^\circ \supseteq (i_n^*)^{-1}(L^\circ)$ is a 0-neighbourhood in \mathcal{T} . Thus $\mathcal{O} = \mathcal{T}$.

(b) By (a), we have $G^* = \varprojlim G_n^*$ as a topological group, whence G^* is reflexive, by Corollary 8.4. Hence also G^{**} is reflexive [6, Proposition 5.9]. Since $i_n^*(G^*)$ is dense in G_n^* for each $n \in \mathbb{N}$, Proposition 8.5 shows that $G^{**} = \varinjlim_{\text{lqc}} G_n^{**}$. Since

$$\kappa := \varinjlim \eta_{G_n} : \varinjlim_{\text{lqc}} G_n \rightarrow \varinjlim_{\text{lqc}} G_n^{**}$$

is a topological isomorphism from G_{lqc} onto G^{**} , we deduce that G_{lqc} is reflexive. To prove the final assertion, note that $\kappa^*: G^{***} \rightarrow (G_{\text{lqc}})^*$ and $\eta_{G^*}: G^* \rightarrow G^{***}$ are topological isomorphisms and hence also $\kappa^* \circ \eta_{G^*}: G^* \rightarrow (G_{\text{lqc}})^*$. By the universal property of γ , the map γ^* is a bijection. A simple calculation shows that $\kappa^* \circ \eta_{G^*} = (\gamma^*)^{-1}$. Hence γ^* is a topological isomorphism.

To complete the proof, note that if $i_{n,m}$ is a topological embedding, then each character on G_m corresponds to a character on $i_{n,m}(G_m)$, which in turn extends to a character of G_n if we assume that $i_{n,m}(G_m)$ is dually closed in G_n . Therefore $i_{n,m}^*$ is surjective in this case and hence also i_n^* . \square

Recall that a topological abelian group G is called *strongly reflexive* if all closed subgroups and all Hausdorff quotient groups of G and G^* are reflexive. For example, every Čech complete nuclear abelian group is strongly reflexive [6, Theorem 20.40]. This enables us to strengthen Corollary 8.4 in the case of nuclear groups:

Proposition 8.8 *The projective limit G of a countable projective system of Čech complete nuclear abelian groups is strongly reflexive.*

Proof. G is Čech complete by [23, Corollary 3.9.9] and nuclear by [9, Proposition 7.7], whence [6, Theorem 20.40] applies. \square

The following proposition will be used to prove strong reflexivity of certain direct limits. We recall that a topological abelian group G is said to be *binuclear* if η_G is surjective and both G and G^* are nuclear [9, p. 152]. The surjectivity of η_G is automatic if G is nuclear and complete (see [25, Theorem 6.4]).

Proposition 8.9 *Let G be a binuclear abelian group which is reflexive and locally k_ω . Then G is strongly reflexive.*

Proof. Since G^* is nuclear and Čech complete, it is strongly reflexive by [6, Theorem 20.40]. Hence also $G \cong G^{**}$ is strongly reflexive. \square

We first deduce a variant of Corollary 4.6 in [4], which was formulated for countable direct sums of strongly reflexive, nuclear abelian k_ω -groups.⁵

Corollary 8.10 *Let $(G_n)_{n \in \mathbb{N}}$ be a sequence of reflexive, binuclear abelian groups which are locally k_ω . Then $\bigoplus_{n \in \mathbb{N}} G_n$ is strongly reflexive.*

Proof. The countable direct sum $G := \bigoplus_{n \in \mathbb{N}} G_n$ is nuclear and reflexive by [9, Propositions 7.8 and 14.11], and locally k_ω by Corollary 5.7. Furthermore, $G^* \cong \prod_{n \in \mathbb{N}} G_n^*$ by [9, Proposition 14.11], whence G^* is nuclear by [9, Proposition 7.6]. Thus Proposition 8.9 applies. \square

The next corollary provides a further generalization.

Corollary 8.11 *Let $((G_n)_{n \in \mathbb{N}}, (i_{n,m})_{n \geq m})$ be a strict direct sequence of reflexive, binuclear abelian groups which are locally k_ω , and $G = \varinjlim G_n$ in the category of Hausdorff topological abelian groups. Then G is strongly reflexive and binuclear.*

Proof. Since G can be realized as a Hausdorff quotient group of the countable direct sum $\bigoplus_{n \in \mathbb{N}} G_n$, it is nuclear by [9, Propositions 7.5 and 7.8] (cf. also [9, Proposition 7.9]). Hence G is locally quasi-convex in particular. Furthermore, G is locally k_ω (by Corollary 5.5) and reflexive, by Proposition 8.7(b). The latter applies because subgroups of nuclear groups are dually embedded [9, Corollary 8.3]. By Proposition 8.7(a), G^* is a projective limit of nuclear groups and hence nuclear (see [9, Proposition 7.7]). Thus G is binuclear and hence strongly reflexive, by Proposition 8.9. \square

Remark 8.12 It would be interesting to find sufficient conditions ensuring that the direct limit Hausdorff topological group $G = \varinjlim G_n$ of an ascending sequence $G_1 \leq G_2 \leq \dots$ of locally quasi-convex abelian k_ω -groups (or locally k_ω groups) is locally quasi-convex. The only available conditions ensuring the local quasi-convexity of a direct limit Hausdorff topological group $G = \varinjlim G_n$ seem to be the following:

- (i) Each G_n is a Hausdorff locally convex space. Or:
- (ii) Each G_n is a nuclear abelian group.

⁵The proof of the cited corollary given in [4] is problematic: Lydia Außenhofer [7] showed by example that the maps $(f_n^m)''$ occurring in its proof need not be embeddings, contrary to claims made there. When we wrote this article, we did not find a way to repair the proof if the G_n 's are merely nuclear (rather than binuclear). After the article was completed, Lydia Außenhofer showed that every nuclear abelian k_ω -group is strongly reflexive and binuclear [8].

Note that G coincides with the direct limit Hausdorff locally convex space in the situation of (i) (cf. [45, Proposition 3.1]), which is locally quasi-convex by [9, Proposition 2.4]). In the situation of (ii), G is nuclear (cf. [9, Proposition 7.9]) and hence locally quasi-convex by [9, Theorem 8.5].

Of course, the direct limit Hausdorff topological G can always be realized as a quotient group of the direct sum $\bigoplus_{n \in \mathbb{N}} G_n$, which is locally quasi-convex, but quotients of locally quasi-convex groups need not be locally quasi-convex [6, Proposition 12.9]. And, as is to be expected, there exist examples of k_ω -groups that are not locally quasi-convex [7], for example the free topological vector space over any non-discrete k_ω -space [30, Proposition 6.4].

In the case of locally convex spaces, no pathologies occur:

Proposition 8.13 *Let $E_1 \subseteq E_2 \subseteq \dots$ be an ascending sequence of locally convex real topological vector spaces E_n , such that the inclusion maps are continuous and linear. If each E_n is a k_ω -space, then the locally convex direct limit topology on $E := \bigcup_{n \in \mathbb{N}} E_n$ turns E into the direct limit topological space $\varinjlim E_n$, and makes it a k_ω -space.*

Proof. It is well known that the box topology on $S := \bigoplus_{n \in \mathbb{N}} E_n$ coincides with the locally convex direct sum topology, and that the locally convex direct limit E can be realized as a quotient of this direct sum by a suitable closed vector subspace. This quotient is also the direct limit $\varinjlim E_n$ in the category of Hausdorff topological groups. Hence the assertions follow from Proposition 5.4. \square

Many locally convex spaces of interest are k_ω -spaces, e.g., all Silva spaces [29, Example 9.4].

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